

CURVE SINGULARITIES AND MONSTER / SEMPLE TOWERS

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ABSTRACT. In earlier work, we introduced the ‘Monster tower’, a tower of fibrations associated to planar curves. We constructed an algorithm for classifying its points with respect to the equivalence relation generated by the action of the contact pseudogroup on the tower. Here, we construct the analogous tower for curves in n -space. (This tower is known as the Semple Bundle in Algebraic Geometry.) The pseudo-group of diffeomorphisms of n -space acts on each level of the extended tower. We take initial steps toward classifying points of this extended Monster tower under this pseudogroup action. Arnol’d’s list of stable simple curve singularities plays a central role in these initial steps. We end with a list of open problems.

1. INTRODUCTION AND HISTORY.

Earlier works [11], [12], [10] constructed a ‘Monster tower’ :

$$\dots \rightarrow M_k \rightarrow M_{k-1} \rightarrow \dots \rightarrow M_1 \rightarrow M_0.$$

of manifolds M_k associated to curves in the plane M_0 . The maps $M_k \rightarrow M_{k-1}$ are fibrations with fiber the projective line. Each M_k is endowed with a rank 2 distribution Δ_k . Here, and throughout, “distribution” means sub-bundle of the tangent bundle. This tower is called the Semple Tower in algebraic geometry (see [10]). The base M_0 can be taken to be any analytic surface in place of the plane. The tower

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is constructed by iterating Cartan's prolongation procedure for distributions as described in [3]. By a 'symmetry' at level k we mean a local analytic diffeomorphism of M_k which maps Δ_k to itself. The distribution Δ_1 is a contact distribution and its symmetries are the contact (pseudo) group. A theorem of Backlund-Yamaguchi ([22]) asserts that all symmetries at level $k > 1$ arise via prolongation from the symmetries at level 1. The central problem addressed in [12] was to classify the orbits of this symmetry group at any level. We largely solved this problem by constructing an algorithm for converting it to the well-studied problem of classifying finite jets of plane curve singularities, and using tools such as the Puiseux characteristic, well-known in that case.

In the present paper we take the first steps towards generalizing this work from the plane to n -space, \mathbb{C}^n . We construct an analogous tower:

$$\dots \rightarrow \mathcal{P}^k(n) \rightarrow \mathcal{P}^{k-1}(n) \rightarrow \dots \mathcal{P}^1(n) \rightarrow \mathcal{P}^0(n) = \mathbb{C}^n.$$

for curves in n -space, \mathbb{C}^n . The fibers are now projective spaces of dimension $n - 1$. When $n = 2$, we have $\mathcal{P}^k(2) = M_k$ of above. When $n > 2$, Backlund-Yamaguchi's theorem now asserts that all symmetries at level k arise from level 0 where the symmetry (pseudo-) group is $Diff(n)$, the pseudo-group of locally defined analytic diffeomorphisms of \mathbb{C}^n . We solve some first occurring instances of the corresponding classification problem by describing "first occurring" orbits that do not arise in the planar case $n = 2$ and by classifying the codimension 1 and codimension 2 singularities at any level, for any n . **We conjecture that the problem of classifying simple stable (with increasing n) singularities of the Monster tower is equivalent to Arnold's classification [1] of simple stable curve singularities.** We verify the first instances of such a correspondence in the course of classifying codimension 1 and 2 singularities for the Monster tower. We also get lower bounds on the number of orbits within $\mathcal{P}^k(3)$, for $k = 2, 3, 4, 5$, indicating many more orbits than in the planar case and discuss relations with the classical Enriques formula as obtained by [10].

The monster tower $\mathcal{P}^k(n)$ is known as the "Semple Tower" in algebraic geometry. It was introduced by Semple ([17]). See in particular the discussion in ([10, 5]). The algebraic geometers typically take the base $\mathcal{P}^0(n)$ of their tower to be n -dimensional projective space, or a general smooth n -dimensional variety, instead of our \mathbb{C}^n .

The tower $\mathcal{P}^k(n)$ is the universal embedding space for the Nash Blow-ups of curves in \mathbb{C}^n . Nash blow up is an alternative to the usual blow up of algebraic geometry, in which the secants lines of the usual blow-up are replaced by tangent lines. (See pp. 412, 3rd paragraph of op. cit. [20] for a history and Nash's original manuscript [15]. (See also [9], esp. pp. 219-221.) In this paper, we lean towards use of the Cartan language so refer to the k -th Nash blow-up of a curve as its " k -th prolongation". The k -th prolongation of a curve in \mathbb{C}^n is an integral curve for Δ_k in $\mathcal{P}^k(n)$. A theorem of Nobile [16] asserts that for sufficiently large k , the k -th prolongation of a singular algebraic or analytic curve is smooth.

Note. We would like to point out that P. Mormul has several papers ([14, 13]) of a kind of parallel nature concerning the classification of points in $\mathcal{P}^k(n)$, $n > 2$, interpreted as flags of special Goursat distributions.

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2. PRELIMINARIES. CONSTRUCTION

All manifolds, curves, maps, etc. are assumed analytic. We will work over \mathbb{C} instead of \mathbb{R} but all results hold for the real case also. As in [10], one could also work over any algebraically closed field of characteristic zero in place of \mathbb{C} .

2.1. Prolongations. Let D be a rank m distribution on a manifold Z . Viewing D as a vector bundle over Z , we form its projectivization $\pi : \mathbb{P}D \rightarrow Z$, the fiber bundle whose fiber over z is the $m - 1$ dimensional projective space $\mathbb{P}(D(z))$. A point of $\mathbb{P}D$ is then a pair (z, ℓ) with $z \in Z$ and $\ell \subset D(z)$ a line in the m -dimensional vector space $D(z) \subset T_z Z$. Define a distribution D^1 on $\mathbb{P}D$ by setting $D^1(z, \ell) = d\pi_{(z, \ell)}^{-1}(\ell)$. Since π is a submersion with $(m - 1)$ -dimensional fibers, and since ℓ is one-dimensional we have $\text{rank}(D) = \text{rank}(D^1) = m$.

Let $\gamma \subset Z$ be a non-constant parameterized integral curve for D . Its prolongation, denoted γ^1 , is the integral curve for D^1 defined by $\gamma^1(t) = (\gamma(t), \text{span}(d\gamma/dt))$ at regular points t for a local parameterization of γ . If $t = t_0$ is not regular, we define $\gamma^1(t_0)$ by taking the limit $\lim_{t \rightarrow t_0} \gamma^1(t)$ of regular points $t \rightarrow t_0$. Our analyticity assumption implies that this limit is well-defined and that the resulting curve $\gamma^1(t)$ is analytic everywhere. For a proof, see [12], pp. 14.

Let ϕ be a symmetry of (Z, D) , meaning a diffeomorphism of Z for which $\phi_* D = D$. Its prolongation, ϕ^1 is the symmetry of $(\mathbb{P}D, D^1)$ defined by $\phi^1(z, \ell) = (\phi(z), d\phi_z(\ell))$. We have $(\phi \circ \gamma)^1 = \phi^1 \circ \gamma^1$ for γ an integral curve for D .

This prolongation construction is due to É. Cartan. We learned of it from R. Bryant ([3]).

* * *

Note. We have described the “rank 1” prolongation of D , “rank 1” in that this prolongation is associated to lines, and so integral curves. For each $r < m = \text{rank}(D)$ there is a “rank r ” prolongation associated to r -dimensional integral submanifolds for D . The $r > 1$ prolongations are significantly more complicated than the $r = 1$ prolongations due to the need to account for the equality of mixed partial derivatives in forming integral submanifolds. Look for “integral elements” in [2] for some details, and see the recent paper by Shibuya-Yamaguchi [19].

2.2. Building the tower. Start with complex n -space \mathbb{C}^n endowed with its tangent bundle $\Delta_0 = T\mathbb{C}^n$ as distribution. Prolong to get the manifold

$$\mathcal{P}^1(n) = \mathbb{P}\Delta_0 \cong \mathbb{C}^n \times \mathbb{C}^{n-1}$$

with distribution Δ_1 . Repeat. After k iterations we obtain a manifold $\mathcal{P}^k(n)$ endowed with the rank n distribution $\Delta_k = (\Delta_{k-1})^1$. Topologically,

$$\mathcal{P}^k(n) = \mathbb{C}^n \times \mathbb{P}^{n-1} \times \dots \rightarrow \mathbb{P}^{n-1}$$

(k copies of projective space), and the projection

$$\mathcal{P}^k(n) \rightarrow \mathcal{P}^{k-1}(n)$$

projects out the last factor.

Definition 2.1. The *Monster tower* for curves in n -space is the sequence of manifolds with distributions, $(\mathcal{P}^k(n), \Delta_k)$, together with the fibrations

$$\dots \rightarrow \mathcal{P}^k(n) \rightarrow \mathcal{P}^{k-1}(n) \rightarrow \dots \rightarrow \mathcal{P}^0(n) = \mathbb{C}^n.$$

We write $\pi_{k,i} : \mathcal{P}^k(n) \rightarrow \mathcal{P}^i(n)$, $i \leq k$ for the projections.

The group of analytic diffeomorphisms of \mathbb{C}^n acts on the Monster tower by prolongation, preserving the levels, as indexed by k , the distributions Δ_k and the fibrations. To avoid restrictions arising from convergence and domain of definition issues and to allow more flexibility in the equivalence relations we will work with pseudogroups and germs instead of globally defined diffeomorphisms. Let $Diff(n)$ be the pseudogroup of analytic diffeomorphisms of \mathbb{C}^n . If $\psi \in Diff(n)$ has domain U and range V then its k -th prolongation ψ^k will have domain $\pi_{k,0}^{-1}(U)$ and range $\pi_{k,0}^{-1}(V)$.

Definition 2.2. We say that two points $p, q \in \mathcal{P}^k(n)$ are equivalent, in symbols $p \sim q$, if there is a diffeomorphism germ $\psi \in Diff(n)$ such that $\psi^k(p) = q$.

Classification Problem. Classify the resulting equivalence classes.

Conceptually, it is often simpler to fix the base points $p_0 = \pi_{k,0}(p)$ and $q_0 = \pi_{k,0}(q)$ to be $0 \in \mathbb{C}^n$. Then we can replace the pseudogroup $Diff(n)$ by the *honest* group $Diff_0(n)$ of germs of diffeomorphisms of \mathbb{C}^n mapping 0 to 0. The classification problem is then replaced by the problem of classifying the orbits for this action on the fiber $\mathcal{P}^k(n)_0 = \pi_{k,0}^{-1}(0) \subset \mathcal{P}^k(n)$ over 0.

3. LANGUAGE. RESULTS.

3.1. The Curve Approach. Take a non-constant curve $\gamma(t)$ in \mathbb{C}^n with $\gamma(0) = 0$. Prolong it repeatedly to form the sequence of curves $\gamma^1, \gamma^2, \dots$ with $\gamma^k(t)$ a curve in $\mathcal{P}^k(n)$, integral for Δ_k and $\pi_{k,i} \circ \gamma^k = \gamma^i$. Since $(\phi \circ \gamma)^k(t) = \phi^k \circ \gamma^k(t)$ for any $\phi \in Diff_0(n)$ we see that

$$(3.1) \quad \text{if } p = \gamma^k(0) \text{ then } \phi^k(p) = (\phi \circ \gamma)^k(0).$$

This observation suggests that we approach our classification problem by turning it into the well-studied classification problem for curve germs. *This curve approach to the problem for the case $n = 2$ was very successful* [12].

Definition 3.1 (preliminary). For $p \in \mathcal{P}^k(n)$, suppose that there is a non-constant curve germ γ in \mathbb{C}^n for which $\gamma^k(0) = p$ and that γ^k is immersed. Then we say that γ realizes p .

Although we have required that γ^k be immersed, it is essential to the definition that we allow γ to be singular.

Let $\tau : (\mathbb{C}, 0) \rightarrow (\mathbb{C}, 0)$ be a non-constant map. Observe that

$$(\gamma \circ \tau)^k(t) = \gamma^k \circ \tau$$

which in particular asserts that if γ realizes p , so does any re-parameterization of γ . In the standard equivalence relation for curves, often called “RL” (for right-left) equivalence we insist that τ is a local diffeomorphism (= reparameterization).

Definition 3.2. Equivalence of curve germs. We will say two curve germs γ, σ in \mathbb{C}^n are equivalent, in symbols $\gamma \sim \sigma$, if and only if there is a diffeomorphism germ $\psi \in \text{Diff}(n)$ and a reparameterization germ $\tau \in \text{Diff}_0(1)$ (thus $\tau(0) = 0; d\tau/dt(0) \neq 0$) of $(\mathbb{C}, 0)$ such that $\sigma = \psi \circ \gamma \circ \tau^{-1}$.

Recall that two points $p, q \in \mathcal{P}^k(n)$ are equivalent, in symbols $p \sim q$, if there is a diffeomorphism germ $\psi \in \text{Diff}(n)$ such that $\psi^k(p) = q$. We have just seen that if p and q are points at the same level, realized by curves γ, σ then $\gamma \sim \sigma$ implies that $p \sim q$. **This fact is the heart of the curve approach.**

From the singularity viewpoint, the simplest of all curve germs are the immersed curves – those with $d\gamma/dt(0) \neq 0$, and they are all equivalent.

Definition 3.3. A point p of the Monster is called a ‘Cartan point’ if it can be realized by an immersed curve germ γ in \mathbb{C}^n .

The following theorem is well-known. It can be found in [22] for example.

Theorem 3.4. *For $k > 1$ the Cartan points of $\mathcal{P}^k(n)$ form a dense open orbit $C^k(n)$ whose complement is a hypersurface. In a neighborhood of any Cartan point the distribution Δ_k is locally diffeomorphic to the canonical distribution on the space $J^k(\mathbb{C}, \mathbb{C}^{n-1})$ of k -jets of analytic maps from \mathbb{C} to \mathbb{C}^{n-1} . Every point of $\mathcal{P}^1(n)$ is a Cartan point.*

We reprove the theorem here, both for completeness, and because in developing the proof we will develop needed tools. Our proof is in two parts, one being the proof of Proposition 3.9 further on in this section, and the other being Example 1 within the subsection of section 4 on KR coordinates.

Definition 3.5. The singular locus at level k is the set $\mathcal{P}^k(n) \setminus C^k(n)$. A singular point is a point of the singular locus.

Theorem 3.4 suggests that we will need singular curves to realize singular points of the Monster. The first occurring singularity in any list of singular curves are the A_k singularities. For even k , these are single-branched, represented as the parameterized curve

$$(3.2) \quad x_1 = t^2, x_2 = t^{2k+1} \quad (\text{ the } A_{2k} \text{ curve })$$

To include curve (3.2) in \mathbb{C}^n , $n > 2$, set the remaining coordinates to zero: $x_3 = \dots = x_n = 0$. Any curve diffeomorphic to a reparameterization of the A_{2k} curve is called ‘an A_{2k} singularity’. Computations done in the next section (‘Example: the A_{2k} singularity’, soon after KR coordinates are introduced) show that for $j \geq k$, the j -fold prolongation of an A_{2k} singularity is immersed.

Theorem 3.6. *The points in $\mathcal{P}^j(n)$ realized by the A_{2k} singularity (eq. 3.2) for $j \geq k > 1$ and any $n \geq 2$ form a smooth quasiprojective hypersurface lying in the singular locus $\mathcal{P}^j(n) \setminus C^j(n)$. The union of these A_{2k} points over this range of k is open and dense within the singular locus at level j .*

We prove this theorem in section 4. To proceed to further results we need some language from singularity theory.

* * *

Synopsis and terminology. A *singularity class* for $\mathcal{P}^k(n)$ is a subset of $\mathcal{P}^k(n)$ which is invariant under the $Diff(n)$ action. If that singularity class forms a subvariety, then its codimension is the usual codimension of this subvariety, as a subvariety of $\mathcal{P}^k(n)$. A point, or a singularity class, is called *simple* if it is contained in a neighborhood which is the union of a finite number of $Diff(n)$ orbits.

The ‘order’ or ‘multiplicity’ of an analytic function germ $f(t) = \sum a_i t^i$ is the smallest integer i such that $a_i \neq 0$. We write $ord(f)$ for this (non-negative) integer. The *multiplicity* of an analytic curve germ $\gamma : (\mathbb{C}, 0) \rightarrow (Z, p)$ in a manifold Z is the minimum of the orders of its coordinate functions $\gamma_i(t)$ relative to any coordinate system vanishing at p . This multiplicity, denoted $mult(\gamma)$ is independent of a parameterization of γ and of choice of a vanishing coordinate system. (Warning: Other uses of the term ‘multiplicity’ applied to curve singularities abound in the singularity theory literature. We are following the use of the word as found in Zariski [23] or Wall [21]. More precisely, every single-branched plane curve singularity $f(x, y) = 0$ with $f(0, 0) = 0$ can be well- parameterized as a curve $\gamma(t) = (x(t), y(t))$ and what we have called the ‘multiplicity of $\gamma(t)$ at $t = 0$ coincides with Zariski’s multiplicity in this case.)

The *embedding dimension* of a curve in the manifold Z is the smallest integer d such that c lies in a smooth d -dimensional manifold $\Sigma \subset Z$. Thus, embedded curves have embedding dimension 1, while A_{2k} curves have embedding dimension 2. A curve germ with embedding dimension at least 3 is called a *spatial* curve.

A curve germ $\gamma : (\mathbb{C}, 0) \rightarrow Z$ is called *simple* if, for all integers N sufficiently large the N -th jet $j^N \gamma$ of γ is contained in a neighborhood which is covered by a finite number of RL-equivalence classes.

We summarize the classification results thus far. Fix the level k_0 . The Cartan points at that level form a single orbit, which is open and dense and whose points are realized by immersed (multiplicity 1) curves. The generic singular points at that level are realized by the A_{2k} singularities for $k \leq k_0$. For each k these realized points form a single orbit of codimension 1. The realizing curves – the A_{2k} singularity – has multiplicity 2 and is strictly planar. In both cases, the representative curves are simple, as are the points they realize. These facts suggest the following refinements to the classification problem.

Refined Classification Questions. Do the codimension 2 singularities within the Monster correspond to multiplicity 3 curves?

Do simple singularities of the Monster always correspond to simple curve singularities?

At what level, and in what codimension, do non-planar singularities first occur?

We answer the first question with theorem 3.23 and the last question in Corollary (Cor. 3.25). Theorem 3.23 and the discussion following it, together with the final section, gives partial answers to the second question.

3.2. Verticality. Baby Monsters. RVT coding. As with any smooth fiber bundle, we have the notion of the ‘vertical space’ for the fibration $\mathcal{P}^k(n) \rightarrow \mathcal{P}^{k-1}(n)$.

Definition 3.7. The vertical space at p is the linear subspace

$$V_k(p) = \ker(d\pi_{k,k-1}(p)) \subset T_p \mathcal{P}^k(n).$$

A vector $v \in T_p \mathcal{P}^k(n)$ or line $\ell \subset T_p \mathcal{P}^k(n)$ is called vertical if $v \in V_k(p)$ or $\ell \subset V_k(p)$.

Since the vertical spaces are the tangent spaces to the fibers of $\pi_{k,k-1}$ and since the symmetry group $Diff(n)$ maps fibers to fibers, we have that symmetries send vertical spaces to vertical spaces. Moreover, the fibers are integral manifolds of the distribution Δ_k , so that

$$V_k(p) \subset \Delta_k(p).$$

Definition 3.8. The point $p = (p_{k-1}, \ell)$ of the Monster at level k , $k > 1$ is called ‘vertical’ if the line ℓ is a vertical line at level $k-1$. Otherwise, that point is called ‘non-vertical’. Every point at level 1 or 0 is considered to be non-vertical.

The following proposition is basic to understanding the Cartan points and the overall structure of the Monster tower.

Proposition 3.9. *Let $p \in \mathcal{P}^k(n)$ and write $p_i = \pi_{k,i}(p)$, $i \leq k$. The point p is Cartan if and only if none of the p_i ’s below p is vertical.*

Proof: Postponed for a few pages.

Note that this proposition contains the bulk of theorem 3.4. All that remains to prove of theorem 3.4 is the assertions regarding equivalence between points of the jet bundle for maps to \mathbb{C}^{n-1} , and that is done in the first example of the next section.

Prolongation can be applied to any manifold F in place of \mathbb{C}^n . We let $\mathcal{P}^0(F) = F$, with its tangent bundle $\Delta_0^F = TF$ as distribution. The prolongation of (F, Δ_0^F) is $\mathcal{P}^1 = \mathbb{P}TF$, with its canonical rank $m = \dim(F)$ distribution $\Delta_1^F = (\Delta_0^F)^1$, etc. Locally $(\mathcal{P}^k(F), \Delta_k^F)$ is analytically diffeomorphic, as a manifold equipped with a distribution, to $(\mathcal{P}^k(m), \Delta_k)$. If (Z, D) is a manifold with distribution and $F \subset Z$ is an integral submanifold of D then $\mathcal{P}^1(F) \subset \mathbb{P}D$, and $\Delta_1^F = D^1 \cap T(\mathcal{P}^1(F))$ over $\mathcal{P}^1(F)$.

We apply these considerations to the fiber $F_k(p) := \pi_{k,k-1}^{-1}(p_{k-1}) \subset \mathcal{P}^k(n)$ through the point p at level k . (As above, $p_{k-1} = \pi_{k,k-1}(p)$.) The fiber is an $(n-1)$ -dimensional integral submanifold for Δ_k . Prolonging, we get $\mathcal{P}^1(F_k(p)) \subset \mathcal{P}^{k+1}(n)$, together with its distribution, $\delta_k^1 = \Delta_1^{F_k(p)}$; that is,

$$\delta_k^1(q) = \Delta_{k+1}(q) \cap T_q(\mathcal{P}^1(F_k(p)))$$

a hyperplane within $\Delta_{k+1}(q)$, for $q \in \mathcal{P}^1(F_k(p))$. Iterating, we obtain embedded submanifolds

$$\mathcal{P}^j(F_k(p)) \subset \mathcal{P}^{k+j}(n),$$

together with hyperplanes $\delta_k^j(q) \subset \Delta_{k+j}(q)$ for $q \in \mathcal{P}^j(F_k(p))$.

Definition 3.10. We call the tower $(\mathcal{P}^j(F_k(p)), \delta_k^j)$, $j = 0, 1, \dots$ the baby Monster through p .

Definition 3.11. A *critical hyperplane* at level k through a point p is any one of the hyperplanes $\delta_i^j(p) \subset \Delta_k(p)$ (with $i + j = k$) associated to the prolongation of a fiber through a point in the tower under p .

A direction $\ell \subset \Delta_k(p)$ is called *regular* if it does not lie in any critical hyperplane. A direction is called *critical* if it does lie in a critical hyperplane, and is called ‘*tangency*’ if that critical hyperplane is not the vertical hyperplane. (A rationale for the tangency terminology can be found in [12].)

A point is called regular, critical, vertical, or tangency, depending on whether the corresponding line one level down is regular, critical, vertical, or tangency.

An integral curve is called ‘regular’ if it is tangent to a regular direction.

Warning: a line can lie in more than one critical hyperplane.

Note the vertical hyperplane $V_k(p)$ is itself a critical hyperplane, being of the form δ_k^0 .

Theorem 3.12. *Through every point there passes a regular integral direction.*

Proof: If p is at level k then there are at most $k - 1$ critical planes through p , one for each point in the tower below p besides p_0 . The complement of a finite collection of hyperplanes is open and dense. Take ℓ to be any line in this complement. Q.E.D.

3.2.1. RC codes and RVT codes. Every point p is either regular or critical. Let p be a point and let $\{p_0, p_1, \dots, p_k = p\}$ be the set of all projections of p to lower levels. (Cf. notation of Proposition 3.9.)

The *RC* code of p is the word $w = w_1 \dots w_k$ of length k in the letters R and C with $w_i = R$ if p_i is regular and $w_i = C$ if p_i is critical. The *RC* class of the word w is the singularity class in $\mathcal{P}^k(n)$, denoted $w \subset \mathcal{P}^k(n)$ by slight abuse of notation, consisting of all those points $p \in \mathcal{P}^k(n)$ having *RC* code w .

Note that $w_1 = R$ for any point, as all points at level 1 are regular.

Example. According to Proposition 3.9, and the spelling rules, the Cartan points at level k are those points whose *RC* code is $R^k = R \dots R$ (k times).

Proposition 3.13. *The codimension of the RC class w is equal to the number of letters w_i which are C ’s. An RC class w adjoins an RC class \tilde{w} – meaning w lies in the closure of \tilde{w} – if and only if \tilde{w} can be made into w by replacing some of the occurrences of the letter C appearing in \tilde{w} by the letter R .*

Proof. The critical planes are hyperplanes within the distribution and the condition that a line lie in a given hyperplane is defined by a single equation. To prove the second assertion, realize that any critical point at any level is the limit of regular lines passing through the same point one level down, and hence \tilde{w} ’s closure contains w . Q.E.D.

Example. If $c(t)$ is the A_{2k} curve, then, we will compute at the end of section 4.2, that the *RVT* code of the point $c^{k+1+s}(0)$ is $R^k C R^s$, corresponding to the fact that the orbit of this curve has codimension 1. It adjoins the Cartan class R^{k+1+s} .

RVT code. Change our alphabet by replacing C by V or T . Set $\omega_i(p) = V$ if p_i is a vertical point, and set $\omega_i(p) = T$ if p_i is a critical point *which is not vertical*. In this way we associate an *RVT* code to each point, and an associated *RVT* class. An *RVT* refinement ω of an *RC* code w is any *RVT* code which becomes w when all occurrences of the letters V and T are replaced by C .

Example. If $w = RCCCCRC$ then the possible RVT refinements of w are $\omega = RVVVRV, RVVTRV, RVTVRV, RVTTRV$.

Spelling rules. As discovered in the book [12], there are only two spelling rules for RVT words $w_1 \dots w_k$. The first rule is that $w_1 = R$: every word starts with R. The second rule is that the letter T cannot immediately follow the letter R, reflecting the fact that for p_i to be a tangency point it must lie in the baby monster of some point in the tower under p_i .

Warning: the class “L”. Critical planes are hyperplanes in an n -dimensional space. The intersection of two such planes will contain a line as soon as $n > 2$, and this line represents a point one level higher. For example, when $n = 3$ the intersection of the vertical plane through p and a critical plane arising from a lower level will contain a line. One is tempted to say that the corresponding point, one level up, is both a V and a T point. We have chosen our terminology so that it is labeled to be a V point, but a special name is useful for such a point. A point $p = (p_k, \ell)$ whose line ℓ satisfies $\ell \in V_k(p) \cap \delta_j^i(p)$ will be called an “L” point. “L” points can be reached by swinging a tangency line $\ell_t \in \delta_j^i$ around until it becomes vertical. Thus an L point lies in the intersection of the closure of a class ending in T with a class ending in a V.

Further refinements of the code are clearly possible, for example by indicating how many critical planes a line lies in, at what level baby monster these planes originate, etc. We leave these further developments to interested parties.

3.3. Points by Curves. We will associate to p the collection $Germ(p)$ of all curve germs which realize it regularly.

Definition 3.14. For p a point at level k , write $Germ(p)$ for the collection of all curve germs $\gamma : (\mathbb{C}, 0) \rightarrow \mathbb{C}^n$ at $t = 0$ whose k -th prolongation γ^k is regular and passes through p : $\gamma^k(0) = p$. From now on, when we say “ γ realizes p ” we mean that $\gamma \in Germ(p)$, so not only is $\gamma^k(0) = p$ and $d\gamma^k/dt|_{t=0} \neq 0$, but also the span of $d\gamma^k/dt|_{t=0} \neq 0$ is a regular direction.

According to theorem 3.12, $Germ(p)$ is non-empty. Since the k -fold prolongation of the k -fold projection of a regular curve is the original curve, we have the following alternative description of $Germ(p)$:

$$Germ(p) = \{\pi_{k,0} \circ \sigma : \sigma \text{ is a regular integral curve germ passing through } p\}.$$

The following proposition is immediate:

Proposition 3.15. *Let σ be a regular integral curve germ at level k .*

- (a) $\sigma^1(t)$ is a regular integral curve germ at level $k + 1$.
- (b) If $\sigma(0)$ is a regular point, then the one-step projection $\pi_{k,k-1} \circ \sigma$ of σ is a regular integral curve germ.

Proof of Proposition 3.9 on Cartan points. We must prove a point is Cartan if and only if its code is R^k . Suppose p is Cartan. Let γ be an immersed curve representing p . Being immersed, all its prolongations $\gamma^1, \gamma^2, \dots$ are immersed and regular, by (a) of the above proposition. Thus all the points in the tower below p are regular and so its code is R^k . Conversely, suppose that all the points in the tower below p are regular. Let $\sigma = \gamma^k$ be a regular curve passing through p and consider its one-step projection $\sigma_1 = \pi_{k,k-1} \circ \sigma$. By (b) of the proposition, and the

fact that p_{k-1} is regular, we have that $\sigma_1 = \gamma^{k-1}$ is regular. Continuing to project we see that all the projections of σ are regular. In particular γ is immersed, and so p is Cartan. Q.E.D.

Later on we will need to know that curves in $Germ(p)$ are well-parameterized.

Definition 3.16. A curve is called well-parameterized if it has a representative for which the map $t \rightarrow \gamma(t)$ is one-to-one.

Equivalently, a curve germ γ is *not* well-parameterized if and only if we can express $\gamma = \sigma \circ \tau$ for some other curve germ $\sigma : (\mathbb{C}, 0) \rightarrow \mathbb{C}^n$ and some *non-invertible* germ $\tau : (\mathbb{C}, 0) \rightarrow (\mathbb{C}, 0)$, i.e. some τ where $d\tau/dt(0) = 0$. See [21] for this fact, and for more details on the notion of well-parameterized.

Lemma 3.17. *If $\gamma \in Germ(p)$ then γ is well-parameterized.*

Proof. Suppose not. Then $\gamma = \sigma \circ \tau$ where $d\tau/dt(0) = 0$ and σ is curve germ. We compute that $\gamma^k = \sigma^k \circ \tau$. It follows that γ^k does not immerse for any k , and so is not regular. Q.E.D.

The following proposition is key to our whole development. In the proposition the statement ' $Germ(p) \sim Germ(q)$ ' means that for any curve $\gamma \in Germ(p)$ there is a $\sigma \in Germ(q)$ with $\gamma \sim \sigma$, and conversely.

Proposition 3.18. *For $p, q \in \mathcal{P}^k(n)$ we have $Germ(p) \sim Germ(q)$ if and only if $p \sim q$.*

Proof. Suppose that $\gamma \in Germ(p)$ is equivalent to $\sigma \in Germ(q)$. Then there is a diffeomorphism $\phi \in Diff(n)$ and a reparameterization $\tau \in Diff_0(1)$ such that $\phi \circ \gamma = \sigma \circ \tau$. Prolonging, and using $\tau(0) = 0$, $\gamma^k(0) = p$, $\sigma^k(0) = q$, we see that $\phi^k(p) = q$. Conversely, suppose that $p \sim q$. Then there is a $\phi \in Diff(n)$ with $\phi^k(p) = q$. Since ϕ^k preserves the distribution it preserves the class of regular curves. Thus, if $\gamma \in Germ(p)$ then $\phi \circ \gamma \in Germ(q)$, showing that $\phi(Germ(p)) \subset Germ(q)$. Using ϕ^{-1} yields $Germ(q) \subset \phi(Germ(p))$. Q.E.D.

Observe that if γ^k is a regular curve then $\gamma^{k+1}(0)$ is a regular point. Consequently, if the RVT class of γ is ω then the RVT class of $\gamma^{k+1}(0)$ is ωR .

Definition 3.19. The R stabilization of an RVT class w is any class of the form wR^q , $q \geq 1$.

We restate Theorem 3.6 in this R-stabilization language.

Theorem 3.20. *Suppose that p is in the R-stabilization of the class R^kV and that $\gamma \in Germ(p)$. Then γ is an A_{2k} singularity.*

The proof of the theorem requires us to develop some tools and is presented at the end of the next section.

The following is an immediate corollary of theorem 3.20 and proposition 3.18,

Corollary 3.21. *Each RVT class R^kVR^m (in any dimension) consists of a single orbit.*

It is worth pointing out a geometric consequence of theorem 3.20

Proposition 3.22. *A point p of type R^kV determines a unique partial flag in \mathbb{C}^n of the form (line, plane) attached at $p_0 \in \mathbb{C}^n$. The line is the tangent line p_1 to any curve $\gamma \in Germ(p)$. The plane is the tangent plane at p_0 to any smooth surface germ containing such a γ .*

Table 1: Codimension two classes

R stabilization of the RVT class	Normal form	Simple?
$R^s VT$	$t^3 e_1 + t^{3s+1} e_2 + O(t^{3s+2})$	Yes
$R^s VV \setminus R^{s+1} VL$	$t^3 e_1 + t^{3s+2} e_2 + O(t^{3s+4})$	Yes
$R^s VR^m V$ $m \geq 1,$	$t^4 e_1 + [t^{4s+2} + t^{4s+2m+1}] e_2 + O(t^{4s+3})$	Yes: $s = 0$. No: $s > 0$
$R^s VL$	$t^4 e_1 + t^{4s+2} e_2 + t^{4s+3} e_3 + O(t^{4s+4})$	Yes: $s = 0$. No: $s > 0$

The point of the proposition is that all A_{2k} curves have embedding dimension 2, and that the tangent plane at p_0 in the proposition is independent of the choice of the particular A_{2k} curve $\gamma \in \text{Germ}(p)$.

We are ready for the next occurring singularities.

3.4. Codimension 2 classes: multiplicity 3 and 4 curves. The codimension two RVT classes are precisely the R -stabilizations of the classes

$$R^s VV, R^s VT; s \geq 1,$$

and

$$R^s VR^m V; m, s \geq 1.$$

We also recall that the closure of $R^s VT$ intersects $R^s VV$ in a class denoted by $R^s VL \subset R^s VV$ whose points (at level $s+2$) correspond to those lines at level $s+1$ which lie in the intersection of the vertical hyperplane and the critical hyperplane born from the previous level. $R^s VL$ has codimension 1 within $R^s VV$.

Following our curve philosophy, these ‘next simplest’ singularities in the Monster should correspond to the “next simplest” curves in the classification schemes for curves. These ‘next simplest’ appear in Gibson-Hobbs’ [8] work on classifying simple space curves. Arnol’d proved that they are stably simple [1]. Arnol’d labels the corresponding multiplicity three classes as $E_{6s+2,p,i}$ and $E_{6s,p,i}$ and various degenerations thereof. (See p. 23, [1].)

Theorem 3.23 (Classification of codimension 2 classes). *In any dimension n the following holds. The R -stabilizations of the classes $R^s VV \setminus R^s VL$ and $R^s VT$ are simple, are realized by curves of multiplicity 3, and their union consists of all points realized by curves of multiplicity 3. The R -stabilizations of $R^s VL$ and of the remaining codimension 2 classes $R^s VR^m V$ are realized by curves of multiplicity 4 and are stable if and only if $s = 1$. (In all these statements $s, m \geq 1$ are integers.)*

3.5. Spatial Classes. Bijection with Arnold-Gibson-Hobbs normal forms.

Adding more R ’s to the critical classes ω of theorem 3.23 has the effect of adding information of higher jets to the corresponding points. The corresponding classes ωR^q will break up into orbits, the stable ones breaking up into finitely many orbits. How many orbits? What are they?

We begin by describing those orbits which cannot be seen in the planar case.

Definition 3.24. A point p is called ‘spatial’ if every curve in $\text{Germ}(p)$ has embedding dimension 3 or greater and at least one of the curves has embedding dimension 3. A point p is called ‘purely spatial’ if every curve in $\text{Germ}(p)$ has embedding dimension 3. A point p is called ‘planar’ if every curve in $\text{Germ}(p)$ has embedding

dimension 2 or more and at least one of the curves has embedding dimension 2. A singularity class is called spatial if all its points are spatial points.

By 3.18 a spatial point cannot be equivalent to a planar point.

Proposition 3.25. *The 1st occurring spatial singularity classes occur at level 3 for $n = 3$. There are two such classes, and each is itself an orbit. One of these classes forms an open subset of RVT (and so has codimension 2) and is realized by the 3rd prolongations of those curves whose 5-jet is (t^3, t^4, t^5) . The other class is RVL (and so has codimension 3) and is realized by the 3rd prolongations of those curves whose 7-jet is equivalent to (t^4, t^6, t^7) .*

The classes of this theorem are simple stable classes, according to theorem 3.23. It is perhaps worth noting that the class RVT , in any dimension greater than 2, decomposes into precisely two orbits, one represented by the spatial curve given above, the other represented by the planar curve (t^3, t^4) .

Theorem 3.26. *Every point of R^sVL is 3-dimensional. Most of the points of the other classes described in theorem 3.23 are spatial. Specifically: there are dense open subsets of R^sVT , of R^sVVR and of R^sVR^mVRR all of whose points are purely spatial.*

Finally, we would like a bijection between the corresponding R -stabilized stable simple classes and the corresponding list of stable simple curves of Arnold and Gibson-Hobbs.

Theorem 3.27. *For the stable classes, $\omega = R^sVT$ and $R^sVV \setminus L$, there is a positive integer q sufficiently large such that the orbits of ωR^q are in bijection with the corresponding stable simple classes of Arnol'd's list starting off with the appropriate Taylor series for that class, as listed in Table 1.*

Example: $\omega = R^3VT$. The class begins (t^3, t^{10}) according to the table. Let us use Arnol'd's notation of $(a, b + c, d)$ to stand for curves with germ $(t^a, t^b + t^c, t^d)$. The representative normal forms starting from $(3, 10)$ are the spatial curves $(3, 10, 11)$, $(3, 10 + 11, 14)$, $(3, 10, 14)$, $(3, 10 + 11, 17)$, $(3, 10 + 14, 17)$, $(3, 10, 17)$ and the planar curves $(3, 10)$, $(3, 10 + 11)$, $(3, 10 + 14)$, $(3, 10 + 17)$. For $q \geq 19$ we are guaranteed that there are precisely 10 orbits within ωR^q , with a point in any orbit being RL -equivalent to one of these 10 germs. (The value of $q = 19$ is a pessimistic upper bound. A value of q of about 8 is sufficient to capture all 10 orbit types.)

The reason behind this theorem is that adding more R 's effectively adds information on the derivatives of curves. Then, by taking q large enough and fixing a point p of ωR^q we have fixed enough of the Taylor series of $\gamma \in \text{Germ}(p)$ so as to be assured which one of the various stable classes it lies in.

4. TOOLS AND PROOFS.

We will need the following lemmas and certain special coordinates called 'KR coordinates' after Kumpera-Ruiz [7]. These lemmas and coordinates will also be essential tools in further sections.

4.1. Properties of regular curves under projection.

Lemma 4.1. *Suppose that Γ is a regular integral curve germ through p . Then its one-step projection $\pi_{k,k-1} \circ \Gamma = \Gamma_1$ is an immersed integral curve. If p is not vertical then its two-step projection, $\Gamma_2 = \pi_{k,k-2} \circ \Gamma$ is immersed. If p is a regular point then Γ_1 is a regular integral curve.*

Proof of lemma 4.1. If Γ is a curve germ on any manifold Z , and π is a submersion of Z onto some other manifold, then $d(\pi \circ \Gamma)/dt = 0$ if and only if $d\Gamma/dt \in \ker(d\pi)$. In our case, Γ is immersed, and not tangent to any critical hyperplane, so in particular, it is not tangent to the vertical hyperplane $\ker(d\pi)$ where $\pi = \pi_{k,k-1}$. Therefore, $d\Gamma_1/dt \neq 0$ and Γ_1 is an immersed curve germ.

Since $\Gamma = \Gamma_1^1$ we have that $\Gamma(0) = \text{span}(d\Gamma_1/dt(0))$, thus, if $p = \Gamma(0)$ is not a vertical point we have that Γ_1 is a non-vertical immersed curve, and the argument of the previous paragraph can be repeated to yield that the two-step projection, Γ_2 is immersed. If $p = \Gamma(0)$ is not critical (not a ‘C’), then $d\Gamma_1/dt$ must span a regular direction, so that Γ_1 is a regular curve. Q.E.D.

Lemma 4.2. *Suppose that the RVT class of p is $\omega_1\omega_2\ldots\omega_k$ and let $i \leq k$ be the last occurrence of the letter V : thus if $\omega_k = V$ then $i = k$ and if $i < k$ we have $\omega_i = V$ while $\omega_j \neq V$ for $j > i$. Let $\gamma \in \text{Germ}(p)$. Then γ^{i-1} is immersed and tangent to the vertical.*

Proof. Apply lemma 4.1 iteratively until we reach level $i - 1$. Q.E.D.

Lemma 4.3. *Let γ be a nonconstant analytic curve germ in \mathbb{C}^n . Then $\gamma^{k+1}(0)$ is a vertical point at level $k + 1$ if and only if $\text{mult}(\gamma^k) < \text{mult}(\gamma^{k-1})$.*

The proof of this lemma requires KR coordinates and so is postponed to the end of the next subsection.

4.2. Kumpera-Ruiz coordinates. KR coordinates for the planar ($n = 2$) Monster were described in detail in [12]. The generalization to general n is straightforward and detailed now. For simplicity of notation we just focus on the case $n = 3$, relegating the general case to a few words near the end of this subsection.

We will write a KR coordinate system for $\mathcal{P}^k(3)$ as $(x, y, z, u_1, v_1, \dots, u_k, v_k)$. The coordinates are such that:

- (1) $\pi_{k,j}(x, y, z, u_1, v_1, \dots, u_k, v_k) = (x, y, z, u_1, v_1, \dots, u_j, v_j)$ is the coordinate representation of the projections $\pi_{k,j} : \mathcal{P}^k(3) \rightarrow \mathcal{P}^j(3)$, for $j \leq k$.
- (2) The last two coordinates u_k, v_k are affine coordinates for the fiber.
- (3) There are 3^k KR coordinate systems covering $\mathcal{P}^k(3)$, corresponding to the 3 affine charts needed to cover each \mathbb{CP}^2 in $\mathcal{P}^k(3) \cong \mathbb{C}^3 \times \mathbb{CP}^2 \times \dots \times \mathbb{CP}^2$ (k times).

We give an inductive construction of the coordinates. beginning with some remarks concerning homogeneous and affine coordinates for projective planes.

4.2.1. Coordinates for a projective plane. Suppose the projective plane to be $\mathbb{P}(E)$, the projectivization of the 3 dimensional vector space E . Suppose that E is endowed with a distinguished ‘vertical’ plane $\Pi_{\text{vert}} \subset E$. Choose linear coordinates $\theta^1, \theta^2, \theta^3$ for E such that $\Pi_{\text{vert}} = \{\theta^1 = 0\}$. The θ^i are a basis for E^* and $[\theta^1 : \theta^2 : \theta^3]$ form homogeneous coordinates on $\mathbb{P}(E)$, sending a line $\ell = \text{span}(v) \in \mathbb{P}(E)$ to the homogeneous triple $[\theta^1(v) : \theta^2(v) : \theta^3(v)] \in \mathbb{CP}^2$. If the line ℓ is not contained in

the vertical hyperplane we have $\theta^1(v) \neq 0$ so we may divide to get standard affine coordinates $u = \theta^2/\theta^1, v = \theta^3/\theta^1$ where we use scaling to write $[\theta^1 : \theta^2 : \theta^3] = [1 : \theta^2/\theta^1 : \theta^3/\theta^1]$. These coordinates cover all of $\mathbb{P}(E)$ except those lines lying in Π_{vert} .

Replace E by a rank-three distribution D over a manifold Z . Take the θ^i to be a local coframe for D . The same formulae and relations hold to yield fiber homogeneous and fiber affine coordinates for the prolongation $\mathbb{P}D \rightarrow Z$ of (Z, D) . We apply these considerations to $(\mathcal{P}^{k+1}(3), \Delta_{k+1})$ the prolongation of $(\mathcal{P}^k(3), \Delta_k)$.

4.2.2. Constructing the KR-coordinates inductively.

The case $k = 1$. Let x, y, z be standard coordinates on \mathbb{C}^3 so that $\{dx, dy, dz\}$ form a coframe for $\Delta_0 = T\mathbb{C}^3$. Consequently $[dx : dy : dz]$ form homogeneous coordinates on $\mathbb{P}(\Delta_0(x, y, z))$ and $(x, y, z, [dx : dy : dz]) : \mathcal{P}^1(3) \rightarrow \mathbb{C}^3 \times \mathbb{CP}^2$ is a global diffeomorphism. There are three corresponding fiber-affine coordinates for $\mathcal{P}^1(3)$, depending on whether $dx \neq 0$, $dy \neq 0$, or $dz \neq 0$. In the case $dx \neq 0$ these coordinates are

$$(4.1) \quad \begin{aligned} u_1 &= dy/dx, \\ v_1 &= dz/dx \end{aligned}$$

We rewrite equations (4.1) as

$$\begin{aligned} dy - u_1 dx &= 0 \\ dz - v_1 dx &= 0 \end{aligned}$$

and these two Pfaffian equations define the distribution Δ_1 on the open set of lines of $\mathcal{P}^1(3)$ for which $dx \neq 0$. A basis for Δ_1^* is formed by the restriction of dx, du_1, dv_1 to Δ_1 .

The case $k = 2$. Let $p_2 = (p_1, \ell) \in \mathcal{P}^2(3)$ project onto a point p_1 lying in our level 1 open set of lines ℓ for which $dx \neq 0$. Then p_1 has KR coordinates (x, y, z, u_1, v_1) . Homogeneous coordinates for the fibers of $\mathcal{P}^2(3) \rightarrow \mathcal{P}^1(3)$ are given by $[dx, du_1, dv_1]$. The vertical hyperplane in Δ_1 is defined by $dx = 0$. We define KR coordinates u_2, v_2 for a neighborhood of p_2 as follows:

- $(u_2, v_2) = (du_1/dx, dv_1/dx)$ if ℓ is not vertical;
- $(u_2, v_2) = (dx/du_1, dv_1/du_1)$ if ℓ is vertical and $du_1 \neq 0$ on ℓ ;
- $(u_2, v_2) = (dx/dv_1, du_1/dv_1)$ if ℓ is vertical and $du_1 = 0$ on ℓ .

From k to $k + 1$: Inductive Hypothesis. Suppose that KR-coordinate systems $\{x, y, z, u_1, v_1, \dots, u_k, v_k\}$ have been constructed near points $p_k \in \mathcal{P}^k(3)$, satisfying conditions (1) and (2) from the beginning of this section. Our inductive hypothesis on the k -th level coordinates is that in each KR coordinate system there is a distinguished ordered triple of coordinates relabeled as (f_1^k, f_2^k, f_3^k) , such that

- (1) (df_1^k, df_2^k, df_3^k) (restricted to Δ_{k-1}) form a basis for Δ_{k-1}^* ;
- (2) two of these three coordinates f_i^k are the fiber affine coordinates $\{u_{k-1}, v_{k-1}\}$ from the previous level;
- (3) $df_1^k \neq 0$ on ℓ where $p_k = (p_{k-1}, \ell)$ and $\ell \subset \Delta_{k-1}(p_{k-1})$;
- (4) $(u_k, v_k) = (df_2^k/df_1^k, df_3^k/df_1^k)$;
- (5) Δ_k is defined by adjoining the Pfaffian equations $df_2^k - u_k df_1^k = 0, df_3^k - u_k df_1^k = 0$ to the Pfaffian equations occurring at the lower levels $j < k$.

Observe that under this hypothesis, a basis for Δ_k^* is df_1^k, du_k, dv_k and that the vertical hyperplane within Δ_k is defined by $df_1^k = 0$.

The inductive step. Take $p_{k+1} = (p_k, \ell) \in \mathcal{P}^{k+1}(3)$ and $\{f_1^k, f_2^k, f_3^k\}$ the ordered triple for p_k at level k . Define $\{f_1^{k+1}, f_2^{k+1}, f_3^{k+1}\}$ by

- $(f_1^k, f_2^k, f_3^k) = (f_1^k, u_k, v_k)$ if $df_1^k \neq 0$ on ℓ , i.e if p_{k+1} is *not* a vertical point;
- $(f_1^k, f_2^k, f_3^k) = (u_k, f_1^k, v_k)$ if $df_1^k = 0$ on ℓ and $du_k \neq 0$ on ℓ ;
- $(f_1^k, f_2^k, f_3^k) = (v_k, f_1^k, u_k)$ if $df_1^k = 0$ on ℓ and $du_k = 0$ on ℓ .

Then we have fiber-affine coordinates at level $k+1$, defined near p_{k+1} by

$$(u_{k+1}, v_{k+1}) = (df_2^{k+1}/df_1^{k+1}, df_3^{k+1}/df_1^{k+1}).$$

One checks without difficulty that $df_1^{k+1}, du_{k+1}, dv_{k+1}$ are a basis for Δ_{k+1}^* and that the Pfaffian system for Δ_{k+1} is obtained by adjoining the equations

$$df_2^{k+1} - u_{k+1}df_1^{k+1} = 0, df_3^{k+1} - v_{k+1}df_1^{k+1} = 0$$

to the equations arising from the lower levels. The inductive hypothesis for the k -th step implies the hypothesis for the $(k+1)$ -th step. We have completed the construction of the KR-coordinate systems.

Example 4.4. At Cartan points KR coordinates are jet coordinates.

Take a Cartan point $p \in C^k(3) \subset \mathcal{P}^k(3)$. Let $p_1 \in \mathcal{P}^1(3)$ be its projection to level 1, a line in \mathbb{C}^3 . Choose linear coordinates x, y, z on \mathbb{C}^3 such that $dx \neq 0$ on p_1 . At each step $j \leq k$ of the KR construction we have $f_1^j = x$, so that $u_k = du_{k-1}/dx, v_k = dv_{k-1}/dx$. Combining equations we get

$$(u_j, v_j) = (d^j y/dx^j, d^j z/dx^j)$$

These are standard jet coordinates for maps from \mathbb{C} to \mathbb{C}^2 . The Pfaffian system for Δ_k is $du_{j-1} - u_j dx = 0, dv_{j-1} - v_j dx = 0, j = 1, \dots, k$ which is the standard distribution on the jet space $J^k(\mathbb{C}, \mathbb{C}^2)$.

In the case of $p \in C^k(n)$ for general dimension n the computations are nearly identical. Take linear coordinates x_1, \dots, x_n on \mathbb{C}^n for which $dx_1 \neq 0$ on the line of p_1 . Set $x = x_1$ and work over the open set at level 1 of those lines for which $dx \neq 0$. At level j the KR-fiber coordinates $u_{2,j}, \dots, u_{n-1,j}$ satisfy $u_{i,j} = d^j(x_i)/dx^j$ in a neighborhood of any curve in $\text{Germ}(p)$. These are standard jet coordinates for maps from \mathbb{C} to \mathbb{C}^{n-1} . The Pfaffian system for Δ_k near p is given by $du_{i,j-1} - u_{i,j} dx = 0, j = 1, \dots, k$, and $i = 2, \dots, n-1$. These define the standard distribution on the jet space $J^k(\mathbb{C}, \mathbb{C}^{n-1})$.

Finishing the Proof of theorem 3.4 . Proposition 3.9 establishes all assertions of the theorem except that concerning the identification of a neighborhood of a Cartan point with a neighborhood of $J^k(\mathbb{C}^1, \mathbb{C}^{n-1})$. This assertion follows directly from the example 4.4 above. Q.E.D.

The construction for KR coordinates on the Monster $\mathcal{P}^k(n)$ for general n proceeds in a nearly identical manner. The main difficulty is *notational*. Label KR coordinates for $\mathcal{P}^k(n)$ as $(x_1, \dots, x_n, u_1^1, \dots, u_{n-1}^1, \dots, u_1^k, \dots, u_{n-1}^k)$. The affine fiber coordinates u_1^k, \dots, u_{n-1}^k are built out of the previous level according to $u_j^k = df_j/dg$ where f_1, \dots, f_{n-1}, g an n -tuple of coordinates selected from among $\{u_1^{k-1}, \dots, u_{n-1}^{k-1}\}$ and one of the coordinates coming from a level less than $k-1$. These coordinates are such that the df_i together with dg form a basis for Δ_{k-1}^* , and $dg \neq 0$ on the line ℓ of $p = (p_{k-1}, \ell)$ which the coordinate neighborhood must cover. We leave further details to the reader.

Proof of Lemma 4.3. The point $\gamma^{k+1}(0)$ is vertical if and only if the curve γ^k is tangent to the vertical at level k . In KR coordinates, γ^k is represented by adjoining $n-1$ new fiber affine coordinates to the KR coordinate representation of γ^{k-1} . The curve γ^k is tangent to the fiber, i.e. to the vertical, if and only if the order of at least one of the new fiber coordinates is less than the orders of *all* of the previous coordinates, those coordinates representing γ^{k-1} . In other words, if and only if $\text{mult}(\gamma^k) < \text{mult}(\gamma^{k-1})$. Q.E.D.

Example 4.5. The A_{2k} singularity. Start with the curve c given by $x = t^2, y = t^{2k+1}$. Then $u_1 = dy/dx = \frac{2k+1}{2}t^{2k-1}$ defines the first prolongation of the curve in KR coordinates (x, y, u_1) . The j th prolongation, $j \leq k$ is given by adding fiber coordinate $u_j = d^j y/dx^j = c_j t^{2(k-j)+1}$ to the previous u_i , $i < j$, where the c_j are positive rational numbers. Consequently, referring to lemma 4.3 and the spelling rules, we see that the first k letters of the RVT code for $\gamma^N(0)$, $N \geq k$, are R 's. At level k , the curve becomes immersed, tangent to the vertical, with lowest order coordinate u_k having order $1 < 2$. It follows that the $(k+1)$ -st letter of the RVT code is V . At level $k+1$ we compute that the new KR coordinate is $dx/du_k = c_{k+1}t$ representing a regular direction, since $c_{k+1} \neq 0$. The curve $c(t)$ regularizes at level $k+1$. Now proposition 3.15 yields that the code of $c^{k+1+s}(0)$ is $R^k V R^s$.

4.3. Preparing Curves. Given a curve germ $\gamma(t)$ in \mathbb{C}^n we can always, by linear change of coordinates, find coordinates x_i centered at $\gamma(0)$ so that when the curve is expressed in these coordinates we have

$$\text{ord}(x_i(t)) < \text{ord}(x_{i+1}(t)).$$

In these coordinates $\text{ord}(x_1(t)) = m$ where $m = \text{mult}(\gamma)$. Finally, we can reparameterize the curve so that $x_1(t) = t^m$. When such coordinates and a parameterization are chosen, we will say we have *prepared* γ . Thus, a prepared curve, is given in these coordinates by $\gamma(t) = (t^m, x_2(t), x_3(t), \dots)$ with the $x_a(t)$ power series in t : $x_a(t) = \sum_{j > m_a} A_{a,j} t^j$, $a = 2, \dots, n$ and $m < m_1 < m_2 < \dots$ etc.

4.4. Proof of the theorem 3.20, the “ A_{2k} theorem”. By example 4.5 immediately above, the A_{2k} singularity realizes a point of type $R^k V R^s$ upon $k+1+s$ prolongations. To finish the proof, we must show that if $q \in R^k V R^s$, and if $\gamma \in \text{Germ}(q)$ then γ is an A_{2k} singularity.

It follows from lemma 4.1 that if q lies one step over p and is a regular point, then $\text{Germ}(q) \subset \text{Germ}(p)$. Iterating, we see that it suffices to show that if $p \in R^k V$ and $\gamma \in \text{Germ}(p)$ then γ has type A_{2k} . It suffices in turn to show that for all $p \in R^k V$ and all $\gamma \in \text{Germ}(p)$, we have $\text{mult}(\gamma) = 2$. This is sufficient because lemma 3.17 tells us that γ is well-parameterized, and any well-parameterized curve γ of multiplicity 2 is an A_{2j} singularity for some j . Finally, we must have $j = k$ since the RVT code of our point $p = \gamma^{k+1}(0)$ is $R^k V$ while the RVT code of a point $\gamma^{k+1}(0)$ for γ an A_{2j} singularity is R^{k+1} if $j > k$ and $R^j V R^{k-j}$ if $j < k$, according to the computation at the end of the last section.

Write $m = \text{mult}(\gamma)$. By definition, γ^{k+1} is a regular integral curve. By lemma 4.1, its one-step projection γ^k is an immersed curve. This immersed curve is tangent to the vertical space, since $\gamma^{k+1}(0)$ is a vertical point. Thus, in a KR coordinate system one of the fiber coordinates, say u , for γ^k has Taylor series $at + \dots$, $a \neq 0$. Since the $\gamma^j(0)$, $j < k+1$ are all regular points, we have, by lemma 4.3, that $\text{mult}(\gamma^j) = m$ for all these $j < k+1$. As discussed just above, we can “prepare”

our curve, which is to say choose coordinates $x = x_1, y = x_2, x_3 \dots, x_n$ so that when $\gamma(t)$ is expressed in these coordinates we have $\text{ord}(x_i(t)) < \text{ord}(x_{i+1}(t))$ and $m = \text{ord}(x(t))$. Write $m_i = \text{ord}(x_i(t))$. The corresponding KR fiber-affine coordinates along the first prolongation of γ are $u_i = dx_i/dx$ while the subsequent fiber-affine KR coordinates at level j , $j < k + 1$ are of the form $d^j x_i/dx^j$ and so have order $m_i - j m$. (Note $\gamma^j(0)$ is a Cartan point for $j \leq k$. See the example above on Cartan points.) It follows that $\text{mult}(x_i) > k m$ for $i \neq 1$. Since γ^k is immersed and tangent to the vertical by lemma 4.2 the coordinate with the next smallest order after $x(t)$, namely $y = x_2$, must have multiplicity $k m + 1$, to yield order $1 = (k m + 1) - k m$ for one of the fiber coordinates at level k . Write this affine coordinate as $u = d^k y/dx^k$. The KR fiber coordinates at level $k + 1$ are then $U = dx/du$ and $V_j = dw_j/du$ where w_j represent the rest of the affine coordinates at level k ($w_j = d^k x_j/dx^k$, $j > 2$).

To finish the proof, we will need to show that the regularity of γ^{k+1} implies that $\text{ord}(U(t)) = 1$. To establish this fact we need to express the critical planes through p in KR coordinates. A basis for $\Delta_{k+1}^*(p)$ is du, dU and the dV_j . U and the V_j coordinatize the fiber through p so that the vertical hyperplane through p is defined by $du = 0$. There is one more critical hyperplane, $\delta_k^1(p)$, through p and this hyperplane arises from the baby monster through p_k , one level down. At level k the fiber coordinates are u and the w_j , hence any curve lying in this fiber has $x = \text{const.}$, and so $dx = 0$. Since $U = dx/du$ we must have that $dU = 0$ along prolongations of curves lying in the fiber through p_k , showing that $\delta_k^1(p)$ is given by $dU = 0$. Thus, the regularity of γ^{k+1} is equivalent to $du \neq 0$ and $dU \neq 0$ along γ^{k+1} . The second condition implies that $\text{mult}(U) = 1$. But $\text{ord}(U) = \text{ord}(x) - \text{ord}(u) = m - 1$, implying that $m = 2$, which is to say, $\text{mult}(\gamma) = 2$. Q.E.D.

Proof of proposition 3.22. Fix a point $p \in R^k V$ and a curve $\gamma \in \text{Germ}(p)$. As per the above proof, there are coordinates x, y, z, \dots , on \mathbb{C}^n associated to γ such that $\gamma = (t^2, t^{2k+1}, 0, \dots) + O(t^{2k+2})$. Moreover, by expressing prolongations of curves near γ in the associated KR coordinates, as per the proof of proposition, we see that any $\bar{\gamma} \in \text{Germ}(p)$, is, after reparameterization, given by $\bar{\gamma} = (t^2, at^{2k+1}, 0, \dots) + O(t^{2k+2})$, with $a \neq 0$. Any surface containing any one of these curves $\bar{\gamma}$ agrees with the surface defined by the equations $z = 0, \dots, x_n = 0$ up to terms in x, y of the form xy, x^{2k} . The tangent space to any such surface is the x, y - plane. Q.E.D.

4.5. Proof of the codimension 2 theorem, theorem 3.23. There are only finitely many RVT classes at any given level, and they exhaust the monster tower at that level. Thus to show a given RVT class ω is simple it suffices to show that

- (a) ω consists of finitely many orbits and
- (b) every class adjacent to ω , i.e. whose closure contains ω , consists of finitely many orbits.

The RVT classes of codimension 2 and length $s + m + 1$ are adjacent to the classes $R^s V R^m$ and R^{s+m+1} . (See proposition 3.13.) We have seen (Theorem 3.4, Proposition 3.9, and Theorem 3.20) that these classes each consist of a single orbit, so criterion (b) holds. It remains to check criterion (a). **We argue class by class**, following the structure of the proof of the A_{2k} theorem 3.20. The argument for every class ω of the theorem begins in the same way.

Suppose that ω is one of the codimension two RVT classes, that $p \in \omega$ and that $\gamma \in \text{Germ}(p)$. Since ω is not a codimension 0 or 1 class, we know that $\text{mult}(\gamma) := m > 2$. We can choose coordinates (x, y, z, \dots) on \mathbb{C}^n and a parameterization so that our curve has the prepared form $\gamma = (x(t), y(t), z(t), \dots) : x = t^m, y = t^{m_2} + o(t^{m_2}), z = \epsilon t^{m_3} + O(t^{m_3+1}), \dots$ with $m < m_2 < m_3 < \dots$ and $\epsilon = 1$ or 0 . Since the first s letters of ω are R 's we know from lemma 4.3 and the spelling rules that the multiplicity of the curves $\gamma, \gamma^1, \dots, \gamma^{s-1}$ are all equal to m , and that their KR -coordinates up to level s are of the Cartan form above, as expressed in example 4.4. In particular, at level s we have fiber affine coordinates

$$u = d^s y / dx^s, v = d^s z / dx^s, \dots$$

Since $\omega_{s+1} = V$ we have, by lemma 4.3, that $\text{ord}(u) < m = \text{ord}(x)$, and because $m_2 < m_3 < \dots$ etc. that $\text{ord}(u) < \text{ord}(v) < \dots$ etc. At order $s+1$ the new fiber affine coordinates are

$$u_2 = dx/du, v_2 = dv/du, \dots \text{ etc.}$$

A basis for $\Delta_{s+1}^*(p_{s+1})$ is du, du_2, dv_2, \dots etc.

There are exactly two critical planes at level $s+1$ and these are given by

- $du = 0$ (the vertical hyperplane) and
- $du_2 = 0$ (the tangency hyperplane δ_s^1).

The class $R^s VT$. Let p be a point in this class and $\gamma \in \text{Germ}(p)$. No letter past the $(s+1)$ -th in the code is a V , so lemma 4.2 implies that γ^s is immersed and tangent to the fiber. Thus, in the notation above, $u = d^s y / dx^s$ has order 1 while the other fiber coordinates at this level have order greater than or equal to 2. Since our point represents $R^s VT$ the immersed curve γ^{s+1} must be tangent to the T hyperplane, so that $du_2 = 0$ along the tangent to γ^{s+1} . This equality simply asserts that $\text{ord}(x) > 2$ since $\text{ord}(u_2) = \text{ord}(x) - \text{ord}(u) = \text{ord}(x) - 1 > 1$. At level $s+2$ the new fiber coordinates are

$$u_3 = du_2/du = d^2 x / du^2, v_3 = dv_2/du = d^2 v / du^2, \dots$$

A basis for Δ_{s+2}^* is du, du_3, dv_3, \dots . The tangency hyperplane δ_s^2 through our level $s+2$ point which corresponds to the baby monster issuing from level s is defined by $du_3 = 0$ while the vertical hyperplane through that point is defined by $du = 0$ again. Since γ^{s+2} is a regular curve, we must have $du_3 \neq 0$ along γ at $t = 0$. Since $u_3 = du_2/du = d^2 x / du^2$ has order $m-2$ we get that $m-2 = 1$ so that $m = 3$.

We now have that $\gamma = (t^3, t^{m_2}, \dots)$. From the expression $u = d^s y / dx^s$ we have that $\text{ord}(u) = m_2 - (3s)$. But $\text{ord}(u) = 1$ so we must have $m_2 = 3s + 1$. Thus $\gamma = (t^3, t^{3s+1}, \dots) + o(t^{3s+1})$. Results from the singularity theory of curves (see e.g. [1]) asserts that any curve whose $(3s+1)$ -jet has this form is simple and is RL equivalent to one of a finite number of curves listed as $E_{6s,p,i}$ in Arnold (op.cit.). All curves in this list are 2 or 3-dimensional.

We have established that γ is simple. By Proposition 3.18, and the adjacency argument (a) above, p is simple. This finishes the proof for the case $R^s VT$ and its R -stabilizations.

The class $R^s VVR^m \setminus R^s VLR^m$. By lemma 4.1 if p is a point in this class and $\gamma \in \text{Germ}(p)$ then γ must immerse at level $s+1$ and is tangent to the vertical at that level. As discussed above, a basis for Δ_{s+1}^* is $\{du, du_2, dv_2, \dots\}$ and in these linear coordinates, the vertical hyperplane within Δ_{s+1} is given by $du = 0$ while

the tangency (non-vertical critical) hyperplane is given by $du_2 = 0$. Since $\gamma^{s+2}(0)$ is a V point, we have that, at $t = 0$, $du = 0$ and since $\gamma^{s+2}(0)$ is *not* a T point (i.e. not an L) point, and γ^{s+1} is immersed, we have that $du_2 \neq 0$. It follows that $\text{ord}(u) > 1$ while $\text{ord}(u_2) = 1$.

At level $s + 2$ we have new KR coordinates

$$u_3 = du/du_2, v_3 = dv_2/du_2, \dots$$

and $\{du_2, du_3, dv_3, \dots\}$ form a basis for Δ_{k+2}^* . In this basis the vertical hyperplane is given by $du_2 = 0$ while the (non-vertical) critical hyperplane δ_{s+1}^1 is given by $du_3 = 0$. Since γ^{s+2} is regular, we have $du \neq 0, du_3 \neq 0$ and so u_2, u_3 both have order 1. But $\text{ord}(u_2) = \text{ord}(x) - \text{ord}(u)$ and $\text{ord}(u_3) = \text{ord}(u) - \text{ord}(u_2) = 2\text{ord}(u) - \text{ord}(x)$. Thus

$$\begin{aligned} \text{ord}(x) - \text{ord}(u) &= 1 \\ 2\text{ord}(u) - \text{ord}(x) &= 1 \end{aligned}$$

The unique solution to this linear system is $\text{ord}(x) = 3, \text{ord}(u) = 2$. Since $\text{ord}(u) = \text{ord}(y) - (s)\text{ord}(x)$ we get $y = t^{3s+2} + O(t^{3s+3})$. We now have γ in the desired normal form, $(t^3, t^{3s+2}, \dots, 0) + O(t^{3(s+1)})$.

Results from the singularity theory of curves (op. cit. see eg [1]) asserts that any curve whose $(3s+2)$ -jet has this form is simple and is RL equivalent to one of the finite number of curves labeled $E_{6s+2,p,i}$ by Arnol'd. See also Gibson-Hobbs [8].

Again, we have established that γ is simple, and by the adjacency arguments that p is simple. We have established that γ has multiplicity 3 and is either a planar or a strictly 3-dimensional curve. This finishes the proof for the case $R^sVV \setminus R^sVL$ and its R-stabilizations.

The class R^sVL . We continue with the notation and coordinates from the previous case. At level $s + 1$, γ^{s+2} is immersed by lemma 4.1 and Δ_{k+2}^* has basis $\{du, du_2, dv_2, \dots\}$, but now $du = du_2 = 0$ along the curve γ^{s+1} since the letter ‘L’ of the code asserts that the curve is tangent to a line lying in both critical planes. It follows that both u and u_2 have multiplicity 2 or greater, while the coordinate of lowest order, v_2 , must have order 1 since the curve is immersed.

At level $s + 2$, we have fiber affine coordinates:

$$u_3 = \frac{du}{dv_2}, v_3 = \frac{du_2}{dv_2},$$

and basis $\{dv_2, du_3, dv_3, \dots\}$ for Δ_{s+2}^* . Through ‘L’ points there are (at least) 3 critical hyperplanes, namely $dv_2 = 0, du_3 = 0$ and $dv_3 = 0$. (When $n = 3$ there are exactly 3 such planes.) More generally, for any n if ω is of the form αRVL where α is arbitrary and $p \in \omega$ then there are exactly three critical planes through p . See the section below on intersection combinatorics of critical planes.) Since γ^{s+2} is regular all three of dv_2, du_3 and dv_3 must be nonzero along the tangent to γ^{s+2} . That is, $\text{ord}(v_3) = \text{ord}(u_3) = 1$. Using $\text{ord}(v_2) = 1$ and the relations defining v_2, u_3, v_3 we derive,

- $\text{ord}(v_2) = 1 = \text{ord}(v) - \text{ord}(u)$,
- $\text{ord}(u_3) = 1 = \text{ord}(u) - \text{ord}(v_2) \Rightarrow \text{ord}(u) = 2$,
- $\text{ord}(v_3) = 1 = \text{ord}(u_2) - \text{ord}(v_2) \Rightarrow \text{ord}(u_2) = 2$.

Putting the results of the second and third equations into the first, we deduce that $\text{ord}(v) = 3$. Since $\text{ord}(u_2) = \text{ord}(x) - \text{ord}(u)$ we obtain $\text{ord}(x) = 4$ and from $\text{ord}(u) = \text{ord}(y) - s \text{ord}(x)$ we derive $\text{ord}(y) = 4s + 2$. From $v = d^s z / dx^s$ we see that $\text{ord}(v) = \text{ord}(z) - s \text{ord}(x)$ so that $\text{ord}(z) = 4s + 3$. Scaling now, we can put our curve into the form $(t^4, t^{4s+2}, t^{4s+3}, \dots) + (0, O(t^{4s+3}), O(t^{4s+4}), \dots)$. A diffeomorphism of the form $(x, y, z, \dots) \rightarrow (x, y + axz, z)$ kills the middle $O(t^{4s+3})$ and puts our curve into the desired form.

According to Gibson-Hobbs, or Arnol'd, the curve above is simple if and only if $s = 1$. This finishes the proof for the case $R^s V L$.

Last case: $R^s V R^m V$. Fiber affine coordinates at level s are u, v, \dots with

$$u = \frac{d^s y}{dx^s}, v = \frac{d^s z}{dx^s}, \dots$$

A basis for Δ_s^* is dx, du, dv, \dots and in these linear coordinates on Δ_s the vertical hyperplane is $dx = 0$. Since the $(s+1)$ -th letter in the code is a V , γ^s is tangent to the vertical: $dx(\gamma^s)'(0) = 0$. Since $\text{ord}(y) < \text{ord}(z)$ we have that the coordinate with lowest order at level s is u . At level $s+1$ fiber coordinates are:

$$u_2 = \frac{dx}{du}, v_2 = \frac{dv}{du},$$

Δ_{s+1}^* has basis du, du_2, dv_2 and in these linear coordinates for Δ_{s+1} the vertical hyperplane is $du = 0$ while the critical hyperplane arising from the baby-Monster one level down is given by $du_2 = 0$. Since $\gamma^{(s+1)}$ is tangent to a regular direction, we have

$$(4.2) \quad \text{ord}(u) = \text{ord}(u_2) < \text{ord}(v_2)$$

From level $(s+2)$, all the way up to level $s+m+1$ the dominant (lowest order) coordinate continues to be u and the subsequent fiber coordinates are derivatives with respect to u :

$$u_3 = \frac{du_2}{du}, v_3 = \frac{dv_2}{du}, \dots, u_{m+2} = \frac{du_{m+1}}{du}, v_{m+2} = \frac{dv_{m+1}}{du}$$

and $\text{ord}(u_i) < \text{ord}(v_i) < \dots$ for $i = 3, 4, \dots, m+2$. We have that γ^{s+m+1} is vertical due to the occurrence of the final V in the RVT code and so u is no longer the dominant coordinate at this level: $\text{ord}(u) > \min\{\text{ord}(u_{m+2}), \text{ord}(v_{m+2})\}$. Lemma 4.1 implies that γ^{s+m+1} is immersed so that u_{m+2} is now the dominant coordinate, with order 1.

At level $(s+m+2)$

$$u_{m+3} = \frac{du}{du_{m+2}}, v_{m+3} = \frac{dv_{m+2}}{du_{m+2}},$$

$$\Delta_{s+m+2}^* = \text{span}\langle du_{m+2}, du_{m+3}, dv_{m+3} \rangle.$$

the *vertical hyperplane* is $du_{m+2} = 0$ and the critical hyperplane arising from the baby Monster one level down is $du_{m+3} = 0$. Since γ^{s+m+2} is regular we have $du_{m+2}, du_{m+3} \neq 0$ along γ^{s+m+2} . Therefore, the order of u_{m+3} is also 1. But

$$1 = \text{ord}(u_{m+3}) = \text{ord}(u) - \text{ord}(u_{m+2}) = \text{ord}(u) - 1 \Rightarrow \text{ord}(u) = 2.$$

Moreover, from equation 4.2 and $u_2 = dx/du$ we deduce $2 = \text{ord}(u_2) = \text{ord}(x) - \text{ord}(u) \Rightarrow \text{ord}(x) = 4$.

In conclusion:

$$\text{ord}(u) = \text{ord}(y) - s \text{ord}(x) \Rightarrow \text{ord}(y) = 4s + 2.$$

Finally, $\text{ord}(z) < \text{ord}(y)$.

Knowing that $x = t^4 + \dots$ and $y = t^{4s+2} + \dots$ we shall determine what is the smallest non-vanishing term in $(t^4, t^{4s+2}, 0) + o(4s+2)$ which makes this germ well-parameterized.

Let us say the first non-vanishing in y is t^N for $N > 4s + 2$, which allows us to write

$$y(t) = t^{4s+2} + d_0 t^N + \dots.$$

By the same arguments as in the beginning of the proof, we have

$$u(t) = \frac{d^s y}{dx^s} = c_1 t^2 + d_1 t^{N-4s} + \dots$$

and d_1 is proportional to the original constant d in the definition of y .

Now either

- $\text{ord}(v) < \text{ord}(x)$ or
- $\text{ord}(v) \geq \text{ord}(x)$.

In the former case, we have $\text{ord}(v_2) < \text{ord}(u_2)$ contradicting equation 4.2 above. Therefore $\text{ord}(v) \geq \text{ord}(x)$. Differentiating this inequality yields $\text{ord}(v_i) > \text{ord}(u_i)$ up to level $s + m + 2$ which justifies why we can safely ignore the ‘ $z(t)$ ’ and other component terms ‘ $(x_i(t), i > 3)$ ’ in the demonstration.

We compute

$$u_2(t) = \frac{dx}{du} = \frac{t^4}{c_1 t^2 + d_1 t^{N-4s} + \dots} = c_2 t^2 + d_2 t^{N-4s} + \dots.$$

It is not hard to convince ourselves that the coefficients c_2, d_2 in this last equation are rational functions of previous Taylor coefficients. Likewise,

$$u_3(t) = \frac{du_2}{du} = c_4 + d_4 t^{N-4s-2} + \dots$$

and moreover c_4 and d_4 are both non-zero. Continuing in this fashion, we can demonstrate that

$$u_{m+2}(t) = c_{m+2} t^{N-4s-2m} + \dots,$$

and $m \geq 1$. From our earlier logic, $u_{m+3} = \frac{du}{du_{m+2}}$ has a non-zero linear part as a function of t , and therefore from

$$u_{m+3}(t) = \frac{d(c_1 t^2 + d_1 t^{N-4s-2} + \dots)}{d(c_{m+2} t^{N-4s-2m} + \dots)},$$

we can conclude, after deriving the resulting quotient series that

$$2 - (N - 4s - 2m) = 1 \Rightarrow N = 4s + 2m + 1.$$

Notice that this latter power is *odd*, guaranteeing that our initial germ is well-parameterized. The parameter d_0 can be rescaled to 1 by RL equivalence and therefore does not correspond to moduli. Also, had we assumed the non-vanishing term t^N appeared in the z -component the resulting curve would be spatial.

The corresponding germs are simple if and only if $s = 1$. We refer to the lists in Gibson-Hobbs and Arnol’d. This finishes the proof for the last case $R^s V R^m V$ and its R stabilizations. Q.E.D.

4.6. Proofs regarding Spatial classes. Proof of 3.25.

The case of RVL has already been proved.

We proceed to the RVT case. Any point p_3 of this type projects one level down to a point of type RV. Fix a point p_2 at level 2 of type RV. All such points are equivalent. As per proposition 3.22, p_2 determines a partial flag passing through p_0 . We choose coordinates (x, y, z) so that p_0 is the origin, the line (p_1) is the x -axis and the plane is x, y - plane. Now, consider the locus of all points p_3 of type RVT lying over p_2 , and consider the set of all resulting curves $\gamma \in \text{Germ}(p_3)$. According to the KR computations done above (the case R^sVT with $s = 1$ within the Proof of the codimension 2 theorem, theorem 3.23), and with the base coordinates being these (x, y, z) , any such curve γ has the form

$$\gamma = (t^3, a_4t^4 + a_5t^5, b_5t^5) + O(t^6), a_4 \neq 0$$

after a reparameterization.

We compute the corresponding fiber KR coordinates

$$\text{level 1 : } u = dy/dx = \frac{4}{3}a_4t + \frac{5}{3}a_5t^2 + O(t^3), v = dz/dx = \frac{4}{3}b_5t + O(t^3);$$

$$\text{level 2 : } u_2 = dx/du = \frac{9}{4a_4}t^2 + O(t^3), v_2 = dv/du = \frac{5}{2a_4}b_5t + O(t^2);$$

$$\text{level 3 : } u_3 = du_2/du = ct + O(t^2), v_3 = dv_2/du = \frac{15}{8a_4}b_5 + O(t).$$

In these coordinates, the 3rd prolongation of γ is $(0, 0, 0; 0, 0; 0, 0; 0, \frac{15}{8a_4}b_5)$. Now γ is spatial if $b_5 \neq 0$. Said invariantly: every curve sharing 5-jet with γ is spatial if and only if $b_5 \neq 0$. We have proved that those points with $b_5 \neq 0$ are purely spatial. Now any curve γ with 5-jet of the given form, having $b_5 \neq 0$ is RL equivalent to (t^3, t^4, t^5) . The proposition is proved. Q.E.D.

Note that the curves with $b_5 = 0$ include the planar curve $(t^3, t^4, 0)$. Their 3rd prolongations forms the rest of the class RVT – the set of points having planar curves in their germ.

4.7. Proof of Theorem 3.26. Looking back at the previous proof, that of proposition 3.25, we observe that the determining factor was whether or not the coefficient b_5 in γ 's Taylor expansion vanished. The point at level 3 in RVT was spatial if and only if the coefficient $b_5 \neq 0$. This coefficient occurs in the 2-jet of the curves γ^1 at level 1. It determines the outcome of points at level 3 = 1 + 2.

To clarify our understanding we use the following lemma. In the lemma we use the symbol $=_{rep}$ to mean “equal up to reparameterization”.

Lemma 4.6. *Let $\Gamma, \tilde{\Gamma} : (\mathbb{R}, 0) \rightarrow (\mathcal{P}^k(n), p)$ be immersed integral curve germs passing through p and $q \geq 1$ an integer. Then $\Gamma^q(0) = \tilde{\Gamma}^q(0)$ if and only if $j^q\Gamma =_{rep} j^q\tilde{\Gamma}$.*

The corresponding map $j^q\Gamma \mapsto \Gamma^q(0)$ from q -jets of immersed integral curves through p to points in the Monster lying q steps over p is algebraic.

The lemma is a generalization from $n = 2$ to general n of a lemma crucial to the book ([12], pp. 54). We give an conceptual proof as an alternative to the book's coordinate-based proof.

Proof of Lemma 4.6. . Forget the distribution Δ_k on $\mathcal{P}^k(n)$ for a moment, treating $\mathcal{P}^k(n)$ as a complex manifold Z . The lemma is certainly true in this more relaxed situation: for immersed curves germs $\Gamma, \tilde{\Gamma}$ in Z , and for $q \geq 1$ we have $\Gamma^q(0) = \tilde{\Gamma}^q(0)$ if and only if $j^q\Gamma =_{rep} j^q\tilde{\Gamma}$. Here, the curves $\Gamma^q, \tilde{\Gamma}^q$ are curves in the q -fold prolongation $\mathcal{P}^q(Z)$ of (Z, TZ) . The lemma is true in this relaxed situation because space $\mathcal{P}^q(Z)$ with its distribution is locally isomorphic to the prolongation tower $\mathcal{P}^q(N)$, $N = \dim(Z)$ with its Δ_q and the points $\Gamma^q(0), \tilde{\Gamma}^q(0)$ are Cartan points in $\mathcal{P}^q(Z)$. Now use the KR computations as per example 4.4 which relate neighborhoods of Cartan points to neighborhoods in the appropriate jet space of curves, thus establishing the relaxed version of the lemma, including the algebraic nature of the map. To finish the proof of the original lemma, simply observe that $\mathcal{P}^{k+q}(n) \subset \mathcal{P}^q(Z)$ is an algebraic submanifold, the curves $\Gamma^q, \tilde{\Gamma}^q$ lie in this submanifold, and their q -jets form an algebraic submanifold of all q -jets of immersed curves through p . Q.E.D.

Case R^sVT . We can almost copy the previous proof. Fix a point p_{s+1} representing R^sV . All such points are equivalent. As per proposition 3.22, p_{s+1} determines a partial flag – a line and a plane – passing through p_0 . Choose coordinates $(x, y, z_1, z_2, \dots, z_{n-2})$ in \mathbb{C}^n centered at p_0 so that the line (p_1) is the x -axis and the plane is x, y – plane. For convenience, set $\mathbf{z} = (z_1, z_2, \dots, z_{n-2}) \in \mathbb{C}^{n-2}$. Consider the locus of all points p_{s+2} of type R^sVT lying over p_{s+1} and the corresponding curve germ set consisting of $\gamma \in \text{Germ}(p_{s+2})$ as p_{s+2} varies over this locus. According to the KR computations above (the case R^sVT within the Proof of the codimension 2 theorem, theorem 3.23) any such curve γ takes the form

$$(4.3) \quad \gamma = (t^3, a_1 t^{3s+1} + a_2 t^{3s+2}, \mathbf{b}_2 t^{3s+2}) + O(t^{3s+3}), a_1 \neq 0$$

after a reparameterization. In this expansion \mathbf{b}_2 is a vector in \mathbb{C}^{n-2} . It plays the role of the scalar b_5 in the previous proof. The s -fold prolongation of γ is immersed, with fiber coordinates of the form

$$u = a_1 n_1 t + a_2 n_2 t^2, \mathbf{v} = \mathbf{b}_2 n_2 t^2 \in \mathbb{C}^{n-2}.$$

where $n_j = n_j(s) = (3s+j)(3(s-1)+j) \dots (3+j)j/3^s$. By the lemma, the 2-jet of γ^s up to reparameterization uniquely determines the point $p_{s+2} = \gamma^{s+2}(0)$. The modifier ‘up to reparameterization’ requires care. Instead, we directly compute: the KR coordinates one step over p_{s+1} are $d^2x/du^2, d^2\mathbf{v}/du^2$ and are given by $(0, \mathbf{b}_2)$ for our curve. We see that the point p_{s+2} is spatial if and only if $\mathbf{b}_2 \neq 0$, which defines an open dense set within R^sVT . All points in this set are realized by a curve equivalent to $(t^3, t^{3s+1}, t^{3s+3}, 0, \dots, 0)$.

Case R^sVVR . Fix a point p_{s+2} representing R^sVV . All such points are equivalent, being represented by a germ RL equivalent to (t^3, t^{3s+2}) . As per proposition 3.22, the one-step projection p_{s+1} of p_{s+2} determines a partial flag – a line and a plane – passing through p_0 . Choose coordinates (x, y, \mathbf{z}) in \mathbb{C}^n centered at p_0 so that the line (p_1) is the x -axis and the plane is x, y – plane. Now, consider the locus of all points p_{s+3} of type R^sVVR lying over p_{s+2} and the corresponding curve germ set of all curves $\gamma \in \text{Germ}(p_{s+3})$ as p_{s+3} varies over this locus. According to the KR computations (the case R^sVV within the Proof of the codimension 2 theorem, theorem 3.23)) any such curve γ takes the form

$$\gamma = (t^3, a_2 t^{3s+2} + a_3 t^{3s+3} + a_4 t^{3s+4}, \mathbf{b}_3 t^{3s+3} + \mathbf{b}_4 t^{3s+4}) + O(t^{3s+5}), a_2 \neq 0$$

after a reparameterization. Note, in this expansion the \mathbf{b}_i are vectors in \mathbb{C}^{n-2} . The non-vanishing of the vector \mathbf{b}_4 will tell us whether or not the point is purely spatial.

The s -fold prolongation of γ has fiber coordinates

$$u = a_2 n_2 t^2 + O(t^4), \mathbf{v} = \mathbf{b}_3 n_3 t^3 + \mathbf{b}_4 n_4 t^4 \in \mathbb{C}^{n-2}.$$

where $n_j = n_j(s) = (3s+j)(3(s-1)+j) \dots (3+j)j/3^s$. The $(s+1)$ -fold prolongation has KR coordinates $u_1 = dx/du, \mathbf{v}_1 = d\mathbf{v}/du$ and so:

$$u_1 = \frac{3}{2a_2 n_2} t + O(t^2), \mathbf{v}_1 = \frac{3}{2a_2 n_2} \mathbf{b}_3 n_3 t + \frac{4}{2a_2 n_2} \mathbf{b}_4 n_4 t^2$$

To simplify our computation, we will first show that we can assume $\mathbf{b}_3 = 0$. Indeed, the value of \mathbf{b}_3 determines the location of the point p_{s+2} in the fiber over p_{s+1} , which as we have mentioned can be placed arbitrarily using a symmetry, all $R^s VV$ points being equivalent. To see this explicitly, note that KR fiber coordinates for the $(s+2)$ -fold prolongation of γ are $u_2 = du/du_1, \mathbf{v}_2 = d\mathbf{v}_1/du_1$. It follows that $\mathbf{v}_2 = c\mathbf{b}_3$ where c is a nonzero constant involving a_2 and the n 's. A diffeomorphism of the form $(x, y, \mathbf{z}) \mapsto (x, y, \mathbf{z} - ax^{s+1})$ kills the term \mathbf{b}_3 in γ and consequently shifts the \mathbf{v}_2 coordinate to zero. Thus, fixing p_{s+2} is tantamount to assuming $\mathbf{b}_3 = 0$.

We now have, fiber coordinates along γ^{s+1} at level $s+1$ of the form

$$u_1 = \frac{3}{2a_2 n_2} t + O(t^2), \mathbf{v}_1 = c\mathbf{b}_4 t^2$$

where, as above c is a nonzero constant involving a 's and n 's. By lemma 4.6, the 2-jet of γ^{s+1} mod reparameterization uniquely determines the point $p_{s+3} = \gamma^{s+3}(0)$. A reparameterization $t \mapsto \lambda t + O(t^2)$ has the effect on the \mathbf{v}_1 term of $c\mathbf{b}_4 t^2 \mapsto c\lambda^2 \mathbf{b}_4 t^2$. In particular this term is nonzero if and only if the original curve γ is equivalent to the curve germ $(t^3, t^{3s+2}, t^{3s+4})$. Consequently, the set of p_{s+3} 's lying over p_{s+2} fall into two types: those for which $\mathbf{b}_4 \neq 0$ and thus are purely spatial, and those for which $\mathbf{b}_4 = 0$ and so their germs contain planar curves (equivalent to $(t^3, t^{3s+2}, 0)$). Because the correspondence (jets) \rightarrow points is algebraic, the locus of these purely spatial points is open and dense.

The case $R^s V R^m V R R$. . We will leave this tedious case to the reader. Q.E.D.

4.7.1. Conclusion: Proof of the Theorem 3.27. We start with the case $R^s VT$. We follow the initial set up and coordinates as with the proof of Theorem 3.26 in subsection 4.7. Thus we fix the point $p_{s+1} \in R^s VT$ and adapted coordinates to p_{s+1} as in that proof. Consider the set of all curve germs γ passing through p_{s+1} such that $\gamma \in p_{s+2+q}$ for some point $p_{s+2+q} \in R^s VTR^q$ lying over p_{s+1} . As per the earlier proof, γ^s is immersed, and γ has the form of equation 4.3. By the lemma, knowing the jet, $j^{2+q}\gamma^s$, up to reparameterization, is equivalent to knowing the point $\gamma^{s+2+q}(0) = p_{s+2+q}$. Once $q \geq 1$, we can fix the parameterization by the insistence that $x = t^3$, since we can 'see' $j^3\gamma$ as part of $j^{q+2}\gamma^s$. The key observation is simply that γ^s includes the information of γ . Once we know $j^{6s+2}\gamma$ we know which of the finite number of singularity classes $E_{6s+2,p,i}$ (and its degenerations) from Arnol'd's list the curve lies in. So, take any $q \geq 6s$. Then a point in $p_{s+2+q} \in R^s VTR^q$ determines the $6s+2$ jet of $\gamma \in \text{Germ}(p_{s+2+q})$, and consequently the particular singularity class. Since every such jet of the given form represents precisely one class, we have established the desired bijection.

The proof for the other case, $R^s VV \setminus R^s VL$ is quite similar and omitted. Q.E.D.

5. DEATH OF THE JET IDENTIFICATION NUMBER. BIRTH OF THE JET SET.

In the book [12] the notion of ‘jet identification number’ was introduced and was a crucial tool in many of the classification and normal form results there. We take a moment to explain why this notion fails in dimension 3 or more, and what might be salvaged out of it.

Recall the symbol $=_{rep}$ means “equal up to reparameterization”.

Definition 5.1. We say that the point $p \in \mathcal{P}^k(n)$ has jet number r if there is a $\beta \in J^r(\mathbb{C}, \mathbb{C}^n)$ such that $Germ(p) = \{\gamma \text{ a curve germ} : j^r \gamma =_{rep} \beta\}$. We call β (mod reparameterization) the jet determining p .

The purpose of the jet identification number is to effectively reduce the size of $Germ(p)$ to a point, namely the r -jet β (mod reparameterization) occurring in the definition. We showed that when $n = 2$ every regular point (point whose code ends in R) has a jet identification number. However, for $n = 3$ points do not have a jet identification number.

Let us see what goes wrong with the jet identification number in three dimensions by dividing the jet identification number definition into two parts.

Definition 5.2 (Jet identification number). There is a unique integer r such that for all $\gamma \in Germ(p)$

- (a) if $\tilde{\gamma} \in Germ(p)$ then $j^r(\gamma) =_{rep} j^r(\tilde{\gamma})$
- (b) if $j^r(\gamma) =_{rep} j^r(\tilde{\gamma})$ then $\tilde{\gamma} \in Germ(p)$.

In dimension $n = 2$ all points with code ending in R, and in particular, the points with code RVR have a jet identification number. We claim any points of type RVR in dimension 3 have no jet identification number.

All such points are equivalent, so, we may as well work with the point $p_* = c^3(0)$ where c is the standard cusp, $(t^2, t^3, 0)$. Consider the deformation of c , given by the family of curves $c_{\alpha,a,b} = (t^2, \alpha t^3 + at^4, bt^4)$. They all represent the class RV . Let’s look at the points $p_{\alpha,a,b} = c_{\alpha,b}^3(0)$ by computing the corresponding fiber KR coordinates

$$\text{level 1 : } u = dy/dx = \frac{3\alpha}{2}t + 2at^2, v = dz/dx = 2bt^2;$$

$$\text{level 2 : } u_2 = dx/du = \frac{4t}{3\alpha} - \frac{16t^2a}{2\alpha^2} + O(3), v_2 = dv/du = \frac{8bt}{3\alpha} - \frac{64abt^2}{9\alpha^2} + O(3);$$

$$\text{level 3 : } u_3 = \frac{8}{9\alpha^2} - \frac{256at}{27\alpha^3} + O(2), v_3 = \frac{16b}{9\alpha^2} - \frac{128abt}{9\alpha^3} + O(2).$$

The point p_* has u_3, v_3 coordinates $8/9, 0$. Fixing the value of the u_3 coordinate to be $8/9$ fixes $\alpha = \pm 1$, and the two values are related by reparameterization. A bit of thought now shows that fixing p_* fixes the 3-jet of c . Thus curves in $Germ(p_*)$ have the same 3-jet up to reparameterization, so by (a) of the definition we must have jet identification number $r \geq 3$. But the curves $c_{0,b}$ have the same 3-jet as $c_{0,0}$ and have $p_{0,b} \neq p_{0,0}$ and so are not in $Germ(p_*)$. By (b) of the definition, the jet identification of p_* cannot be 3. On the other hand, since $p_{\alpha,0} = p_{0,0}$ there are curves with different 4-jet from $c_{0,0}$ but still lying in $Germ(p_*)$, which shows by

(a) that the jet-identification of p_* cannot be 4. The jet identification cannot be greater than 4 since item (b) of the definition holds for $r = 4$ and any $r \geq 4$.

We conjecture that with the exception of Cartan points, there are no points in $\mathcal{P}^k(n)$, $k > 1, n > 2$ with a jet identification number.

To try to rescue something from the jet identification number in dimension 3 and higher we observe that points in $Germ(p_*)$ in the above example are characterized by $\alpha = \pm 1, b = 0$ but a arbitrary: the three-jet is determined, modulo reparameterization and *part* of the 4-jet. Instead of a jet identification number, we get a *jet set*, together with a relevant *interval* of jet numbers, here 3 and 4.

Definition 5.3 (Jet interval). Fix a point p of the monster. Suppose that there are integers r, R such that for all $\gamma \in Germ(p)$

- (a) if $\tilde{\gamma} \in Germ(p)$ then $j^r(\gamma) =_{rep} j^r(\tilde{\gamma})$
- (b) if $j^R(\gamma) =_{rep} j^R(\tilde{\gamma})$ then $\tilde{\gamma} \in Germ(p)$. The maximum of the integers r will be denoted by r_1 . The minimum of the integers R will be denoted by r_2 . If $r_1 \leq r_2$ then we call $[r_1, r_2]$ the jet interval.

We note in that if an integer r exists as per item (a), then any integer less than r also works in (a). And that if an integer R works as in item (b) of the definition, then any integer greater than R works in (b). We believe that $r_1 \leq r_2$ is a consequence of the definitions.

In the case of the cusp above, $[r_1, r_2] = [3, 4]$.

6. INTERSECTION COMBINATORICS OF CRITICAL PLANES.

We say that a collection $\Lambda_i, i \in I$ of linear hyperplanes in a vector space is “in general position” if, for every subset $J \subset I$ of indices whose cardinality is less than or equal to the dimension of the vector space we have that the codimension of the subspace $\bigcap_{i \in J} \Lambda_i$ is equal to the cardinality $|J|$ of J .

Theorem 6.1. *The critical planes through $p \in \mathcal{P}^k(n)$ are in general position within the vector space $\Delta_k(p)$ and there are at most n of them.*

Recall the following language since it will be central to the proof. The baby Monster originating from $p_k \in \mathcal{P}^k(n)$ is the tower of submanifolds with induced distributions $(\mathcal{P}^j(F_k(p)), \delta_k^j) \subset (\mathcal{P}^{k+j}(n), \Delta_{k+j})$ obtained by prolonging the fiber $F_k = \pi_{k,k-1}^{-1}(p_{k-1}) \subset \mathcal{P}^k(n)$ through p_k at level k .

The proof of theorem 6.1, by induction on the ambient dimension n , will rely on

Proposition 6.2. *Within the baby Monster, $\mathcal{P}^j(F_k(p))$ the critical planes are all of the form $\delta_k^j \cap \delta_r^s$ with $r > k$ and $r + s = k + j$.*

Proof of the proposition. For $c > 0$ the fiber of $\mathcal{P}^c(F_k(p)) \rightarrow \mathcal{P}^{c-1}(F_k(p))$ is $F_{k+c} \cap \mathcal{P}^c(F_k(p))$. It follows that the baby monsters *within* the baby Monster originating from p_k are obtained by intersecting p_k ’s baby Monster with those originating at higher levels. Taking tangents yields the proposition.

We will also be relying on the following linear algebra lemma, stated without proof.

Lemma 6.3. *If $\Lambda_0, \Lambda_1, \dots, \Lambda_s$ is a collection of linear hyperplanes in V such that collection $\Lambda_0 \cap \Lambda_1, \dots, \Lambda_0 \cap \Lambda_s$ is in general position within Λ_0 then the original collection is in general position within V*

Proof of theorem 6.1. By induction on n . The base of the induction is the case $n = 2$ of the theorem, for curves in the plane and was proved in the book [12] where it was central to the development.

The inductive hypothesis is *hypothesis* H_n : The critical planes through a point of $\mathcal{P}^k(n)$ are in general position and are less than or equal to n in number. Assuming H_{n-1} , we will prove H_n .

To this end let $p \in \mathcal{P}^k(n)$. Let $k_0 \leq k$ be the smallest integer such that a critical hyperplane $\delta_{k_0}^{j_0}$ originating from level k_0 passes through p . (Necessarily $j_0 + k_0 = j + k$.) Then the successive levels of the baby Monster arising from level k_0 pass through successive points p_{k_0+i} , $i = 0, 1, \dots, k - k_0$ in the tower under p . By proposition 6.2, the critical planes of this baby Monster are of the form $\delta_{k_0}^i \cap \delta_r^s$, $r > k_0$. By the inductive hypothesis H_{n-1} applied to the $(n-1)$ -manifold F_k we have that these planes are in general position and there are no more than $n-1$ of them. By the linear algebra lemma 6.3 the collection $\delta_{k_0}^i$ together with δ_r^s is in general position and there are no more than n of them. Q.E.D.

Theorem 6.1 asserts that there are at most n critical planes passing through any point. But we can often do better than this. Through a Cartan point there passes exactly one critical hyperplane, namely the vertical hyperplane. Through an A_{2k} point, i.e. one with code R^kV there pass two critical planes, the vertical hyperplane, and the critical hyperplane arising from the baby monster one level down. To get a bound in the general situation let $p \in \mathcal{P}^k(n)$ let $\omega = \omega_1 \dots \omega_k$ be its RC code.

Definition 6.4. The length of the critical tail for p is equal to zero if $\omega_k = R$. If $\omega_k = C$ then this length is the number of consecutive letters ending with ω_k which are C's

Proposition 6.5. *The number of critical planes through p is less than or equal to 1 plus the length of the critical tail through p .*

Proof. Follow how the critical planes through p can arise out of baby monsters. Observe that if a projection p_i of p is a regular point, (that is, the corresponding line one level down is a regular direction) then no critical planes can arise out of the baby monster one level down, or further, since these baby monsters cannot pass through p_i , it being a regular point. Q.E.D.

6.1. RVL is non-planar. We give two applications of the incidence relations for critical planes. Recall that we proved that any curve representing the class R^sVLR^q is of the form $(t^4, t^{4s+1}, t^{4s+2}, \dots) + O(t^{4s+4})$. This jet is that of a non-planar curve: there is no smooth surface which contains the curve near $t = 0$. It follows that no point in this RVT class can be touched by a planar curve. We give an alternative 'synthetic' proof of the non-planarity of these points, a proof which simultaneously establishes the non-planarity of points in any RVT class having the letter 'L'. We then generalize this theorem to higher-dimensional curves.

Theorem 6.6. *Let $\zeta = (p, \ell)$ be a point of the monster at level $k+1$ for which the line ℓ lies in the intersection of two critical planes through the point p at level k . Then ζ cannot be touched by the prolongation of a planar curve.*

Proof. It suffices to prove that if γ is planar then its prolongations γ^k are never tangent to a line lying in two critical planes. Since the condition of planarity and the condition of being a line contained in two critical planes are diffeomorphism invariant conditions, we may use symmetry considerations, and take the surface containing γ to be the xy-plane F_0 sitting inside \mathbb{R}^3 .

We can now work entirely in the $n = 3$ context. The trick is to treat F_0 in the same way as we treated fibers in the baby monster construction, thinking of it as a “level 0 fiber”, and prolonging it so as to obtain the “level 0 baby monsters”, the submanifolds-with-distribution $(\mathcal{P}^k(F_0), \delta_0^k)$ within $\mathcal{P}^k(3)$. We have that $\delta_0^k \subset \Delta_k$ is a 2-plane to which γ^k is tangent. Throw the δ_0^k in to the collection of critical hyperplanes and observe that the resulting larger collection of critical hyperplanes is still in general position, as is seen by going back over the proofs of Theorem 6.1 and Proposition 6.2, viewing F_0 as another fiber. Now, a given line ℓ in a 3-space can be contained in at most 2 two-planes out of a collection of two-planes which are in general position. Our line ℓ , the tangent line to γ^k , lies in δ_0^k since $\gamma \subset F_0$. It follows that ℓ can lie in at most one other critical hyperplane. Q.E.D.

Recall that a curve has embedding dimension d if it lies in a smooth d -dimensional submanifold.

A nearly identical proof to that just given yields,

Theorem 6.7. *Let $\zeta = (p, \ell)$ be a point of the monster for which the line ℓ lies in the intersection of $d - 1$ critical hyperplanes through the point p . Then ζ cannot be touched by the prolongation of a curve with embedding dimension less than d .*

This is a special instance of E. Casas-Alvero’s projection theorem in the language of the prolongation tower. See ([4], especially pp. 318).

6.2. Orbit counts in the spatial case. The introduction of the class ‘L’ allows us to further split the RVT classes into many more classes. Theorem 6.1 asserts that at most three critical planes can pass through a point when $n = 3$. The maximum of 3 is realized if and only if the point is of ‘L’ type. Then its three critical planes are in general position, and each pair gives rise to another ‘L’ direction, and hence a new type of L point one level up. Therefore we have 3 directions of type ‘L’, and two distinct types of tangency directions since their corresponding baby monsters are born in different levels. An immediate conclusion of this simple reasoning is that there are at least 7 geometrically distinct types of directions passing through a ‘L’ point : regular, vertical, 2 types of tangency, and 3 types of ‘L’ directions. Adding together these contributions level-by-level we compute a *lower bound* for the number of orbits in the spatial case $n = 3$ summarized in table 6.2. The tree graph in figure 1 summarizes this ‘branching’ of geometric classes in first four levels of the extended monster tower.

7. SEMPLE TOWER = MONSTER TOWER

Algebraic geometers, beginning with Semple ([17]) in the 1950s, have been working on the Monster Tower, which they now call the Semple Tower. See ([5]) and ([10]). Semple’s original tower concerned the planar case, and has base \mathbb{P}^2 rather than our \mathbb{C}^2 .

M. Lejeune-Jalabert makes a particularly beautiful use of the Tower in ([10]) in order to generalize F. F. Enriques ([6]) famous formula relating the multiplicities

level	planar orbits(sharp)	spatial orbits
first	1	1
second	2	2
third	5	6
fourth	13	23
fifth	34	98

TABLE 1. Orbit counting comparison between planar and spatial cases. The planar number are exact. In the spatial case the numbers are lower bounds.

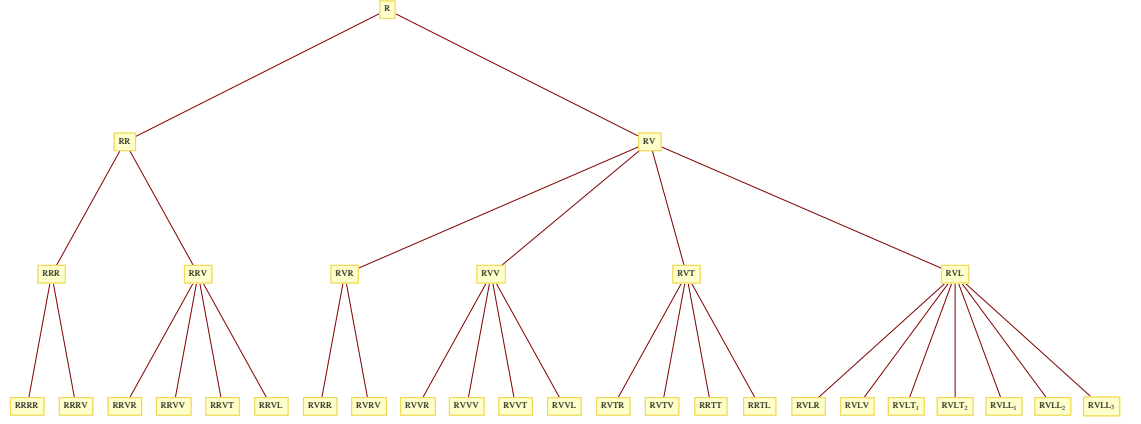


FIGURE 1. Classification of geometric orbits in $P^k(3)$ for $k \leq 4$.

of points on consecutive blow-ups of singular plane curves. M. Lejeune-Jalabert generalized this formula to the case of curves in \mathbb{C}^n . We tested our development of the Monster against her results to obtain an alternative derivation of her Enriques' formula.

Theorem 7.1 (Enriques). *Given a germ $\gamma : (0, \mathbb{C}) \rightarrow (0, \mathbb{C}^n)$, let $S = \{\gamma^i(0)\}_{i=0}^\infty = \{p_i\}$. Then,*

$$\text{multiplicity of } p_i = \sum_{p_j \text{ proximate to } p_i} \text{multiplicity of } p_j$$

The statement of the theorem uses the notion of points at different levels being *proximate*. *Proximity* is a classical notion in the algebraic theory of curves, applied to sequences of ‘infinitely near’ points obtained from classical blow-ups of the curve. Semple ([18] or [?]), and then Lejeune altered the notion so as to fit the Nash blow-up (= prolongation) of the curve as it fits within the Semple tower. The following definition is equivalent to Lejeune’s (op. cit. [?]).

Definition 7.2. A point $p \in \mathcal{P}^k(n)$ is said to be *proximate* to a point q , $q = \pi_{k,j}(p)$ under p in the tower if either:

- (1) $j = k - 1$
- (2) there a vertical curve σ through q whose prolongation (sufficiently many times) passes through p .

From our structure theorem on the critical hyperplanes through a given point, any point in $p \in \mathcal{P}^k(n)$ is proximate to at most n points. (Sitting at lower levels.)

By keeping track of the multiplicity of consecutive prolongations of a well-parameterized singular germ γ one can show that after sufficiently many prolongations the prolonged curve is a *regular germ*. This fact yields an alternative proof of the following theorem of Nobile [16]:

Theorem 7.3 (Nobile; Castro). *Every well-parameterized curve germ has a finite regularization level.*

We shall present our demonstration in a different note.

8. OPEN PROBLEMS.

We end with some open problems. Throughout $p \in \mathcal{P}^k(n)$ and γ is a non-constant well-parameterized curve germ in \mathbb{C}^n .

8.0.1. On simplicity.

Q1. : Is p is tower simple if and only if every $\gamma \in \text{Germ}(p)$ is simple?

Q2. : Is γ is simple if and only if all the points $\gamma^j(0)$ are simple? There are also

stable versions of these two questions. To formulate them, we use the embeddings $\mathbb{C}^n \rightarrow \mathbb{C}^{n+1}$ to obtain embeddings $\mathcal{P}^k(n) \rightarrow \mathcal{P}^k(n+1)$ which take distribution into distribution. The following diagram (“the Russian doll”) may be helpful:

$$\begin{array}{ccccccc}
 & \downarrow & & \downarrow & & \cdots & \downarrow & \cdots \\
 \mathcal{P}^k(2) & \longrightarrow & \mathcal{P}^k(3) & \longrightarrow & \cdots & \longrightarrow & \mathcal{P}^k(n) & \longrightarrow \cdots \\
 & \downarrow & & \downarrow & & \cdots & \downarrow & \cdots \\
 & \vdots & & \vdots & & \vdots & \vdots & \cdots \\
 & \downarrow & & \downarrow & & \cdots & \downarrow & \cdots \\
 \mathcal{P}^1(2) & \longrightarrow & \mathcal{P}^1(3) & \longrightarrow & \cdots & \longrightarrow & \mathcal{P}^1(n) & \longrightarrow \cdots \\
 & \downarrow & & \downarrow & & \cdots & \downarrow & \cdots \\
 \mathbb{A}^2 & \longrightarrow & \mathbb{A}^3 & \longrightarrow & \cdots & \longrightarrow & \mathbb{A}^n & \longrightarrow \cdots
 \end{array}$$

Arnol’d used the standard embeddings $\mathbb{C}^n \rightarrow \mathbb{C}^{n+1}$ to define the notion of a “stably simple curve singularity”: one which is simple independent of the embedding dimension. He listed all these stably simple curves.

Q3: Does iterated prolongation induce a bijection between Arnol’d’s stably simple curves and the R-stabilizations of stably simple points?

8.0.2. On discrete invariants attached to points.

Discrete invariants.

Q4. : For $p \in \mathcal{P}^k(n)$ is it true that $\text{mult}(\gamma)$ is independent of the choice of $\gamma \in \text{Germ}(p)$?

If so, we would call this number the ‘multiplicity of p ’. (If not we could take the minimum of the multiplicities, but that would be less satisfactory.)

Instead of the multiplicity of a curve, we could take any discrete invariant $\Lambda(\gamma)$ of curve germs and ask, for given p is the value $\Lambda(\gamma)$ constant for all $\gamma \in \text{Germ}(p)$? If ‘yes’ we would then say that the invariant Λ is well-defined for p . Possible invariants are the semi-group and the parameterization number. Again, if the invariant is not well-defined, but sits within a partially ordered set, perhaps we can take a minimum of its values on $\gamma \in \text{Germ}(p)$ to get an invariant of the point. In addressing these questions, and the ones around them, some substitute for the jet identification number, so useful in dimension 2, but debunked in dimension 3 and greater, will be of great help.

Q5. : Does every regular point (one whose RVT code ends in R) in $\mathcal{P}^k(n)$, $n > 2$ have a jet interval, with associated jet set, as defined above?

8.0.3. *Curve-to-Point philosophy. Continuity of the prolongation-evaluation map.* Write $\text{Germ}_w^* \subset \text{Germ}^*$ denote the space of all *well-parameterized* curve germs $(\mathbb{C}, 0) \rightarrow (\mathbb{C}^n, 0)$. We have explicitly excluded the constant curve. Write \mathcal{P}_0 for the fiber of the infinite Monster over $0 \in \mathbb{C}^n$, this being the direct limit as $k \rightarrow \infty$ of the $\mathcal{P}^k(n)$. A point of \mathcal{P}_0 is an infinite sequence $(0, p_1, p_2, \dots, p_i, p_{i+1}, \dots)$ with $p_k \in \mathcal{P}^k(n)$ and $\pi_{k,i}(p_k) = p_i$. We have the prolongation map

$$\text{Prol} : \text{Germ}_w^* \rightarrow \mathcal{P}_0$$

by sending γ to $\text{Prol}(\gamma) = (\gamma(0), \gamma^1(0), \dots, \gamma^k(0), \dots)$. By theorem 7.3, γ^i is regular for i sufficiently large, so that all the $\gamma^i(0)$ are regular points. It follows that the range of \mathcal{P}_0 is contained in the subset \mathcal{P}_0^R of points which are eventually regular.

A coordinate computation with immersed curve germs shows that the image of Prol is not all of \mathcal{P}_0^R : some kind of “growth” conditions on KR coordinates are also required if the domain of Prol consists of analytic functions. For consider the coefficients b_j of the coordinate functions x_a along the curve. Being analytic functions, these coefficients satisfy a bound $|b_j| \leq C^j$ for some constant C depending only on γ . Preparing the coordinates and parameterization so $x(t) = t$, the KR coordinates of $\text{Prol}(\gamma)$ at level j , written (u_j, v_j, \dots) , have the form are of the form $d^j x_a / dx^j$, etc, and so satisfy the bounds $|u_j| \leq j! C^j$.

Q6. : What is the image of Prol ?

Let us denote this image by $\mathcal{P}_0^{an} \subset \mathcal{P}_0^R$, the ‘an’ being for analytic.

It seems quite impossible to make Prol into a continuous map relative to any reasonable topology on the space of curve germs. To see the problem, take N large and consider the curve $\gamma = (x(t), y(t)) = (t^N, t^{N+1})$. Then $\gamma^1(0) = (0, 0, 0)$ in standard coordinates where the last coordinate represents dy/dx . But, for any $A \in \mathbb{C}$, and $r < N$ we can find an arbitrarily C^r -small perturbation $\tilde{\gamma}$ of γ with $\tilde{\gamma}^1(0) = (0, 0, A)$. However, restricted to curves of multiplicity 1 (immersed curves) Prol is beautifully continuous.

Q7. : Is there some discrete curve invariant Λ such as multiplicity, such that $Prol$ is continuous when restricted to the class of all curves of constant Λ ?

As a kind of converse to the previous question we ask .

Q8. : If γ is a well-parameterized curve germ with γ^j regular, then, is there an $r \geq j$ such that a neighborhood base for $\gamma^j(0) \in \mathcal{P}^j(n)$ induces a C^r neighborhood base the curve germ γ ? An affirmative answer would yield a potentially powerful technical and conceptual tool.

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